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September 24-27, 2014, Ohrid, R. Macedonia

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HEURISTIC TOOLS FOR THE GENERATION OF NEW MATHEMATICAL FACTS

Sava Grozdev[1]

Abstract. Some geometric configurations are considered about circles and related generalizations are proposed including second-order curves, mainly conics. Several approaches are discussed to compose new mathematical facts: to generalize some initial conditions, generalizations of the problem solution under consideration, etc. Computer animations are also productive and examples are presented. The paper refers to the last several years IMO Geometry problems, which are of high quality and content, proposing possibilities for further investigations and generalizations. In a corresponding research process accompanying results appear in a natural way. The software program "THE GEOMETER'S SKETCHPAD" (GSP) is applied as a heuristic tool for the purpose.

1. INTRODUCTION

During the last decade a new trend is developing in Mathematics Education, namely the *Synergetics approach*. Basic concepts and ideas in this direction are discussed in [1] and [2]. Practical applications are developed in the frames of the International project MITE (Methodology and Information Technology in Education) with the participation of the Moscow State University "M. Lomonosov", the Academy of Social Control in Moscow region, the Federal University in Arhangelsk, the Institute of Mathematics and Informatics with the Bulgarian Academy of Sciences and other institutions. The monograph [3] appeared recently as a result of the productive collaboration among the participating partners of the project. A new partner is Skopje, represented by Prof. Dr. R. Malcheski, Dr. V. Gogovska, Mag. K. Anevska and others. One of the basic domains in the Synergetics approach is the so called Mathematics by Computer, where a new computer program "Discoverer" is applied, based on Artificial Intelligence principles ([4], [5] and [6]). New interesting mathematical facts are obtained in Geometry, including results by THE GEOMETER'S SKETCHPAD software as heuristic tool, mainly in [7] and [8]. Our work is inspired by the words of Rene Descartes (1596–1650) at the dawn of contemporary Science: "*More we go into the Abstract, more we need concrete examples and experiments*." And also by the words of Andre Fouche:

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"*Intuition comes later, after the first successful experiments. Intuition is a result of the successes and not their reason.*"

2. A PROBLEM FOR THE ORTHOCENTER OF A TRIANGLE

The following problem from the 49-th IMO'2010 paper is considered:

Problem 1. *Given is an acute ΔABC with orthocenter H. The circle through H, whose center is the midpoint of the side BC, meets the sideline BC at A_1 and A_2. Analogously, the circle through H, whose center is the midpoint of the side CA, meets the sideline CA at B_1 and B_2. Also, the circle through H, whose center is the midpoint of the side AB, meets the sideline AB at C_1 and C_2. Prove that the points A_1, A_2, B_1, B_2, C_1 and C_2 are co-cyclic.* (Fig. 1)

Consider an arbitrary ΔABC, for which the points M_a, M_b and M_c are the midpoints of the sides BC, CA and AB, respectively.

3. CURVES, THAT ARE GENERATED BY AN ARBITRARY POINT IN THE PLANE OF A TRIANGLE

According to Problem 1, the points A_1, A_2, B_1, B_2, C_1 and C_2 are co-cyclic. Note that the circle is a special second-order curve. Now replace the point H by an arbitrary point P in the plane of ΔABC. Using a similar construction as in Problem 1, we get 6 points A_1, A_2, B_1, B_2, C_1 and C_2, which induce the following theorem (still a hypothesis for the moment) (Fig. 2):

Theorem 1. (analogous to Problem 1) *Given is a triangle ABC and an arbitrary point P in its plane. The circle through P, whose center is the midpoint of the side BC, meets the sideline BC at A_1 and A_2. Analogously, the circle through P, whose center is the midpoint of the side CA, meets the sideline CA at B_1 and B_2. Also, the circle through P, whose center is the midpoint of the side AB, meets the sideline AB at C_1 and C_2. Then, the points A_1, A_2, B_1, B_2, C_1 and C_2 lie on a second-order curve $k(P)$, which turns out to be a circle iff P is the orthocenter of ΔABC.* (Fig. 2)

In what follows the notation $k(X)$ will be used for a second-order curve with center X (if the curve is central). Additionally, experiments with GSP suggest the following:

Corollary 1. *If* P *is on the circumcircle* Γ *of* ΔABC *, then* $k(P)$ *is a hyperbola or it degenerates into two perpendicular lines (the set of the two lines is a special second-order curve).*

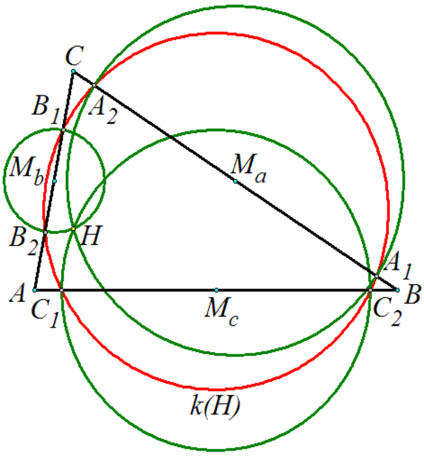

Figure 1

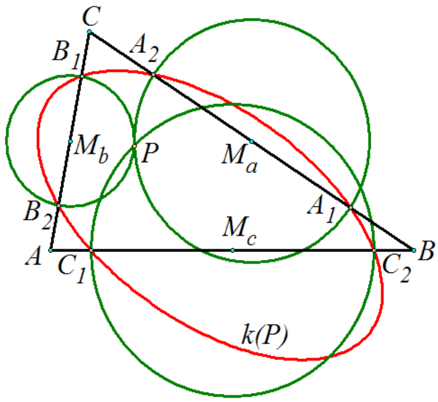

Figure 2

4. Curves, that are generated by isogonal conjugate points with respect to a triangle

Note that the midpoints M_a, M_b and M_c, which are circle centers in Problem 1, are the orthogonal projections of the circumcenter O onto the sidelines of ΔABC, while the common point of the three circles is the orthocenter H. Let us turn upside down the situation, i. e. let the orthogonal projections of the orthocenter H be centers of the three circles passing through the circumcenter O. Now, the corresponding 6 points on the sidelines are co-cyclic. Something more, the GSP experiment shows that the new circle coincides with the previous one. Since H and O are isogonal conjugate with respect to ΔABC, then one could consider any other isogonal conjugate pair. We come to another assertion (again still a hypothesis for the moment):

Theorem 2. *Let* P^1 *and* P^2 *be isogonal conjugate points with respect to a given* ΔABC*, while the points* P_a^j*,* P_b^j *and* P_c^j *be the orthogonal projections of* P^j $(j=1,2)$ *onto the sidelines* BC*,* CA *and* AB*, respectively. The circle through* P^s $(j \neq s = 1,2)$ *with center* P_a^j *meets the sideline* BC *in* A_1^j *and* A_2^j*. The pairs* B_1^j*,*

B_2^j and C_1^j, C_2^j are defined on the sidelines CA and AB respectively in a similar way. Then, the points A_1^j, A_2^j, B_1^j, B_2^j, C_1^j and C_2^j $(j=1,2)$ lie on equal circles with centers P^1 and P^2. (Fig. 3)

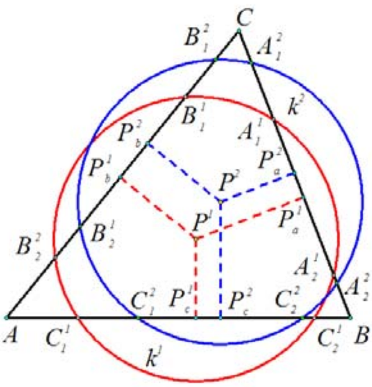

Figure 3

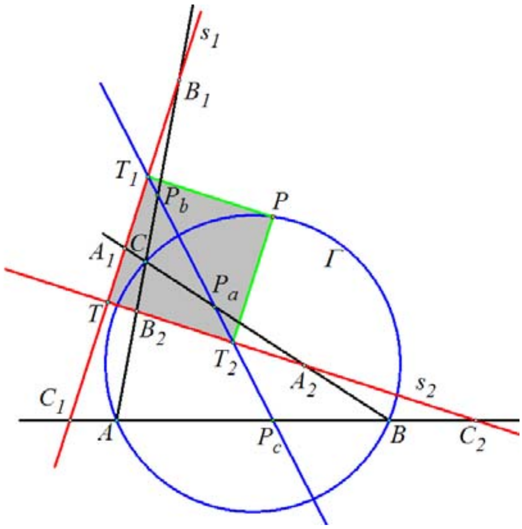

Figure 4

Of course, the points on the circumcircle Γ of ΔABC (and only they) do not satisfy Theorem 2, since they have no isogonal conjugate ones with respect to ΔABC in the sense of finite points. In this case the "defect" could be removed as in the Corollary 1. Construct the circles through $P \in \Gamma$ with centers P_a, P_b and P_c. Consider the intersection points of the circles with the sidelines BC, CA and AB in the way it is accomplished in Problem 1, Theorem 1 and Theorem 2. We state the following:

Theorem 3. *Let P be on the circumcircle Γ of ΔABC, while P_a, P_b and P_c be its orthogonal projections onto the sidelines BC, CA and AB, respectively. The circle through P with center P_a meets the sideline BC in A_1 and A_2. Analogously, the circle through P, with center P_b meets the sideline CA in B_1 and B_2. Also, the circle through P with center P_c meets the sideline AB in C_1 and C_2. Then, the points A_1, A_2, B_1, B_2, C_1 and C_2 lie on two perpendicular lines s_1 and s_2.* (Fig. 4)

It is well-known that the points P_a, P_b and P_c from Theorem 3 are collinear and the line s_P on which they lie is known to be the Simson line. In connection with the Simson line note the following result:

Corollary 2. *The intersection points of* s_1, s_2 *and* s_P *together with the generating point* P *define the vertices of a square.* (Fig. 4)

5. PROOFS

We have to legalize the formulated assertions in this paragraph by mathematical proofs. It is enough to limit ourselves to Theorem 1, since the applied technique is the same. Details could be found in the book [8]. Barycentric coordinates will be used with regard to ΔABC, namely $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$. Let $|BC|=a$, $|CA|=b$ and $|AB|=c$. Then

$$16S^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4,$$

where S is the area of ΔABC. For an arbitrary point $P(\lambda,\mu,\nu)$ $(\lambda+\mu+\nu=1)$ in the plane of ΔABC consider the notation: $\delta = a^2\mu\nu + b^2\nu\lambda + c^2\lambda\mu$. The point P lies on the circumcircle of ΔABC iff $\delta = 0$. Remind also that if $M_1(x_1,y_1,z_1)$ and $M_2(x_2,y_2,z_2)$ are points in the plane of ΔABC, then:

$$|M_1M_2|^2 = -(y_1-y_2)(z_1-z_2)a^2 - (z_1-z_2)(x_1-x_2)b^2 - (x_1-x_2)(y_1-y_2)c^2.$$

If the vectors $\overrightarrow{u_1}(\lambda_1,\mu_1,\nu_1)$ and $\overrightarrow{u_2}(\lambda_2,\mu_2,\nu_2)$ $(\lambda_1+\mu_1+\nu_1=0,\ \lambda_2+\mu_2+\nu_2=0)$ are coplanar to the plane of ΔABC, then they are perpendicular ($\overrightarrow{u_1} \perp \overrightarrow{u_2}$) iff the following equality is verified:

$$(\mu_1\nu_2+\mu_2\nu_1)a^2 + (\nu_1\lambda_2+\nu_2\lambda_1)b^2 + (\lambda_1\mu_2+\lambda_2\mu_1)c^2 = 0.$$

Proof of Theorem 1. It is clear that the theorem is meaningless when P coincides with one of the vertices A, B and C. Such cases are excluded.

Let $p_a = \frac{|PM_a|}{a}$, $p_b = \frac{|PM_b|}{b}$ and $p_c = \frac{|PM_c|}{c}$. The coordinates of the 6 points under consideration are deduced from above:

$$A_1(0,\tfrac{1}{2}+p_a,\tfrac{1}{2}-p_a),\ A_2(0,\tfrac{1}{2}-p_a,\tfrac{1}{2}+p_a),\ B_1(\tfrac{1}{2}-p_b,0,\tfrac{1}{2}+p_b),$$
$$B_2(\tfrac{1}{2}+p_b,0,\tfrac{1}{2}-p_b),\ C_1(\tfrac{1}{2}+p_c,\tfrac{1}{2}-p_c,0),\ C_2(\tfrac{1}{2}-p_c,\tfrac{1}{2}+p_c,0).$$

Substitute the coordinates and check the following equation:

$$k(P):\ \begin{aligned}&(4p_a^2-1)(4p_b^2-1)(4p_c^2-1)(x^2+y^2+z^2) + 2(4p_b^2-1)(4p_c^2-1)(4p_a^2+1)yz + \\ &+2(4p_c^2-1)(4p_a^2-1)(4p_b^2+1)zx + 2(4p_a^2-1)(4p_b^2-1)(4p_c^2+1)xy = 0.\end{aligned}$$

This proves that the points A_1, A_2, B_1, B_2, C_1 and C_2 lie on a second-order curve $k(P)$.

Remark. In the proof it is not used the fact that the circles under consideration are concurrent at a point. This suggests a possibility to examine particular cases: for example the cases when they are tangent to a circle, tangent to a line or cases with other additional conditions.

It follows that:

$$|M_aP|^2 = a^2p_a^2 = \tfrac{1}{2}(-a^2+b^2+c^2)\lambda + \tfrac{1}{4}a^2 - \delta,$$
$$|M_bP|^2 = b^2p_b^2 = \tfrac{1}{2}(a^2-b^2+c^2)\mu + \tfrac{1}{4}b^2 - \delta,$$
$$|M_cP|^2 = c^2p_c^2 = \tfrac{1}{2}(a^2+b^2-c^2)\nu + \tfrac{1}{4}c^2 - \delta.$$

In order to prove the second part of Theorem 1, denote by O_A and O_C the centers of the circumcircles of the triangles $A_1A_2B_1$ and $C_1C_2B_2$, respectively. Further, find the equations of two perpendicular bisectors of each triangle and consider the corresponding system. Thus, the coordinates of O_A and O_C could be determined in the following way:

$$x_{O_A} = -\frac{2a^2[4p_{ab}+2(-a^2+b^2+c^2)p_b-c^2]}{32S^2(1-2p_b)},$$
$$y_{O_A} = \frac{4(a^2+b^2-c^2)p_{ab}-4b^2(a^2-b^2+c^2)p_b+2a^2b^2+b^2c^2+c^2a^2-a^4-b^4}{32S^2(1-2p_b)},$$
$$z_{O_A} = \frac{4(a^2-b^2+c^2)p_{ab}-4c^2(a^2+b^2-c^2)p_b+2a^2b^2+3b^2c^2+c^2a^2-a^4-b^4-2c^4}{32S^2(1-2p_b)},$$
$$x_{O_C} = \frac{4(a^2-b^2+c^2)p_{cb}-4a^2(-a^2+b^2+c^2)p_b+2b^2c^2+3a^2b^2+c^2a^2-b^4-c^4-2a^4}{32S^2(1-2p_b)},$$
$$y_{O_C} = \frac{4(-a^2+b^2+c^2)p_{cb}-4b^2(a^2-b^2+c^2)p_b+a^2b^2+2b^2c^2+c^2a^2-b^4-c^4}{32S^2(1-2p_b)},$$
$$z_{O_C} = -\frac{2c^2[4p_{cb}+2(a^2+b^2-c^2)p_b-a^2]}{32S^2(1-2p_b)},$$

where $p_{ab} = a^2p_a^2 - b^2p_b^2$ and $p_{cb} = c^2p_c^2 - b^2p_b^2$.

If the circles coincide, then $O_A \equiv O_C$ and we deduce that:

$$-2a^2(-a^2+b^2+c^2)\lambda + (3a^2-b^2+c^2)(a^2-b^2+c^2)\mu - (a^2-b^2+c^2)(a^2+b^2-c^2)\nu = 0,$$
$$(-a^2+b^2+c^2)(a^2+b^2-c^2)\lambda + 2(c^2-a^2)(a^2-b^2+c^2)\mu - (a^2+b^2-c^2)(-a^2+b^2+c^2)\nu = 0.$$

Add the condition $\lambda+\mu+\nu=1$, thus obtaining a linear system of three equations with three unknowns:

$$\lambda_H = \frac{(a^2-b^2+c^2)(a^2+b^2-c^2)}{16S^2},$$
$$\mu_H = \frac{(a^2+b^2-c^2)(-a^2+b^2+c^2)}{16S^2},,$$
$$\nu_H = \frac{(-a^2+b^2+c^2)(a^2-b^2+c^2)}{16S^2}.$$

The result coincides with the coordinate representation of the orthocenter H. Finally, note that the unique solution of the system, describing the relation $O_A \equiv O_C$, implies that H is unique, so that $k(P \equiv H)$ is a circle. This ends the proof.

Further details are included in the book [8].

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ONE BINARY LESSON

Rumjana Angelova

Abstract. The article discusses one binary lesson of mathematics and physics with students from X-th grade of Vocational School of Economics and Management – Pazardzhik, Bulgaria, it presents a preparation, describes the implementation, considers the difficulties, analyzes the results. Mathematical ideas and structures and the problem of radioactive decay are considered in synthesis.

1. INTRODUCTION

The binary lessons are one of the forms of realization of interdisciplinary communication and integration items. This is an unconventional view of the lesson. These lessons allow students to integrate knowledge from different fields to solve a problems, provide an opportunity to apply their knowledge in practice, to communicate simultaneously with two teachers. One possibility to solve the issue of students' motivation and out of the traditional template lesson is the application of binary lesson. The emphasis of these lessons is placed on strengthening emotional and personal significance of education, putting active, dynamic overall experience of the students.

As part of an integrated educational technology the binary are lessons, the most interesting, based on interdisciplinary connections, as it implies the use of an alloy of different pedagogical techniques. The purpose of the binary lesson - to create conditions motivating practical application of knowledge, skills and give students the opportunity to see the results of their work, and give them joy and satisfaction.

2. ONE BINARY LESSON

The goal of the lesson was to reach all our pupils and help them to use their full potential, by adapting the activities to their own skills and strengths. The lesson was organized and conducted by two teachers. The duration of the lesson was 80 minutes /40+40/, because when teachers work together can inspire their students successfully, can clearly see gaps in their knowledge.
When we organize binary lessons we have specific pedagogical technology.
The careful planning of the lesson was an important stage in its preparation.

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Key words and phrases. Binary lesson, pedagogical technology

The binary lesson has implemented many of the principles of learning, but the priorities are identifying and developing:

a) The conceptual framework;

b) The content of the training;

- Learning Objectives - general and specific;
- The content of the training material;

c) The procedural part - the manufacturing process;

- The organization of the educational process;
- Methods and forms of learning activities students;
- Methods and forms of teachers;
- Activity of the teachers to manage the process of mastering the material;
- Activity of the students;
- Diagnosis of the educational process.

Through the organization and conduct of binary lesson the teachers were better able to deploy inquiry-based learning, because it is not about using new tasks. It is not about using practical experiments. It is of course true that the tasks and materials must offer students the opportunity to make decisions for themselves, but the tasks and materials do not in themselves guarantee inquiry-based learning. It is rather a perspective on learning that creates a new learning culture in the classroom [1].

In a study and output of the Law of radioactive decay, the math teacher skillfully guided the work of the students. On the worksheets they drew and recorded quantities which are obtained after each period of decay, numerical sequences of the quantities. They built the graphic in easier way, illustrating the law. The teachers provoked students, showing the graphic through the motions of their body [dynamic math]. This motivated them to actively participate in the course of the lesson.

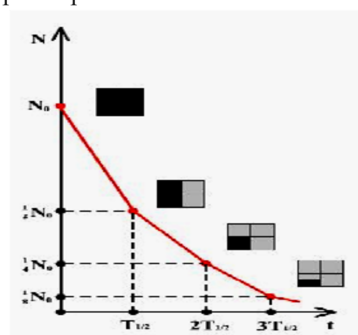


Figure 1: Law of radioactive decay

The roles of the teachers were dynamically changing. The Intellectual cooperation of both educators with the students during the lesson determined the depth of the utilized learning material [3].

Mathematical ideas and structures and the problem of radioactive decay were considered in synthesis.

An important moment when we organize binary lessons is preparing and giving pre-tasks. After a web-research, some students presented results for the biggest stone field in the village of Dorkovo and suggested math tasks with the usage of Carbon-14 dating.

Figure 2: Pliocent Park, Dorkovo

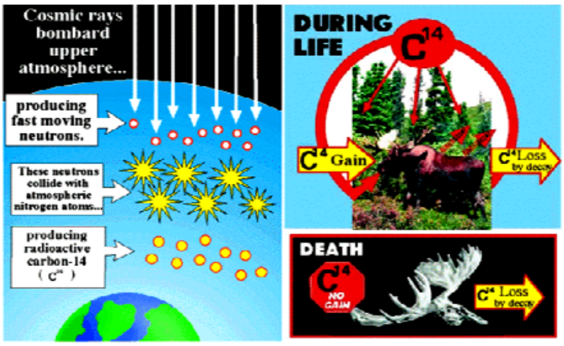


Figure 3: Carbon - 14 dating

During the binary lesson the following has been done:

- Development of cooperation of teachers;
- Formation of students' belief in the connection objects in the integrity of the world.

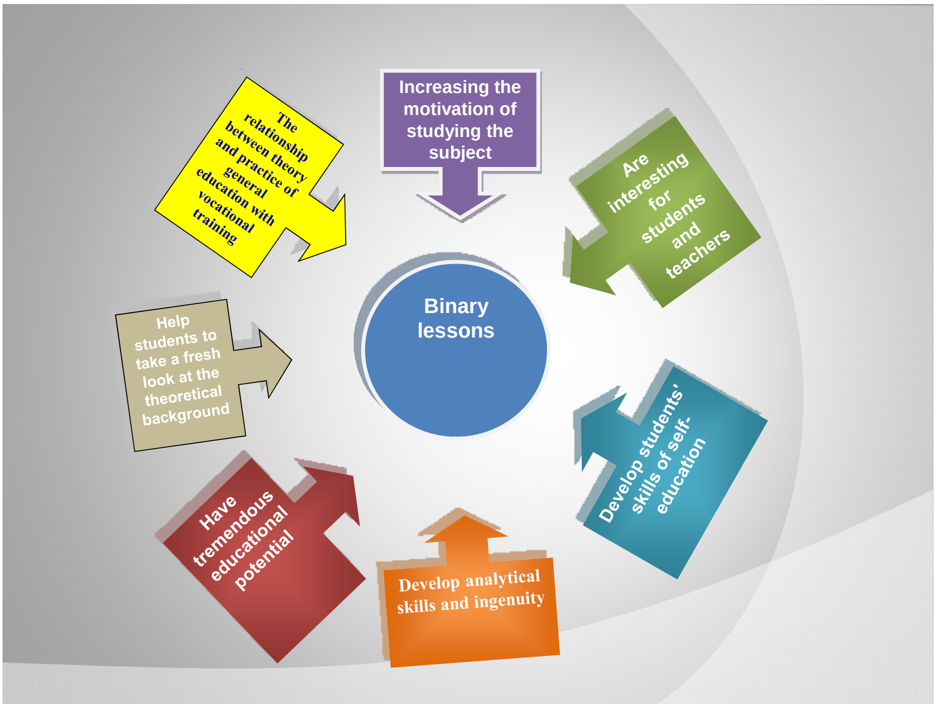
During the binary lesson we organized a variety of activities:

1. Task "Young archaeologist": Determining the age of ancient archaeological sites of biological origin;
2. Consideration and visualization of numerical sequences, functional dependencies;
3. Compilation of mathematical models in solving problems of radioactive decay;
4. WEB searching;
5. Group work.

The feedback of the inquiry showed that:

- Students from the other classes have told "We also want! and they started "attacks" ;
- 56% of the students told their parents about the lesson;
- 89% would like to participate again in a binary lesson;

- 22% of the students drew up tasks for radioactive decay and carbon dating and sent them to a data base;
  Students suggested binary lessons mathematics and accounting, mathematics and economics, mathematics and chemistry,… and biology,…and art.



The Students identified the lesson as:

- interesting, entertaining, useful;
- full of a variety of activities;
- working together with two teachers inspired them.

The Students determined that, the eviction of the law of radioactive decay and the compilation of mathematical models of the tasks, is much more clear through the guidance of the teacher of mathematics and working with numerical sequences and graphics of the function.

What difficulties did we meet?

1. In the curricula there is only an indication of a general nature and the possibility of using specific data from other sciences is not shown.
2. There is no overall coordination of the curriculum in mathematics, economics, accounting, physics, chemistry, art etc.
3. 3 The technological organization and conducting of the binary lessons are not enough developed.

4. The organization is not an easy process, it is necessary to be carefully considered and coordinate at any time.

3. FURTHER PERSPECTIVES

We will keep working in the following areas:

- Organizing and conducting binary lessons;
- Describe the methodology;
- Organizing the survey for the results of the binary lessons;
- Sharing experiences;
- Creating a model of binary lesson as pedagogical technology.

We also focus on the organization and study of binary lessons between mathematics and professional economic subjects, because the vocational training in its structure is binary, two-component. Properly organized vocational training must report the necessity of the unity of the theoretical and practical training and assure it. In other words, vocational training must be binary not only by its primary structure (presence and close connection of the theory and practice), but also technical (the real process of training - both theoretical knowledge and professional competences to be formed inseparably, and to assure, in an organic connection, a binary effect which will be ready for a professional activity [3].

Tasks-scenarios must be discussed as potential resources [1].

4. CONCLUSION

By conducting binary lessons we will encourage students to think critically and creatively, to improve their autonomy and teamwork, to achieve teaching style in which "Infusion of knowledge" gives way to sharing and joint searches in scientific-didactical spirit [2]. With this type of activity and engagement, students are inspired to obtain a deeper knowledge of the subjects they study.

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METHODICAL APPROACH FOR INTRODUCTION OF EXPONENTIAL ENTRY OF COMPLEX NUMBER IN SECONDARY EDUCATION

Katerina Anevska

Abstract. During the study of complex numbers, students in secondary education adopt the algebraic and trigonometric entry of complex number. However, many nonstandard geometric problems can be solved using complex numbers, where the so called exponential entry of complex number and the Euler formulae have the most important role , which are not studied in secondary education. In this paper a methodical approach will be given which will justify the introduction of exponential entry of complex number and Euler formulae and their application in study of Euclidean geometry by using complex numbers

1. INTRODUCTION

Adopting the term complex number, as well as the elementary application of complex numbers are part of the aims of study in secondary school. However, since its complexity and because of lack of knowledge of other parts of mathematics, a part of knowledge which is necessary for the development of the students, especially the gifted students are not included in the curricula of mathematics in secondary education. Such is the case with exponential entry of complex number and Euler formulae, which are actually the relation between the exponential and trigonometric functions.

In mathematic the trigonometric functions $\sin$ and $\cos$, have a particular place, which can be introduced with no use of geometrical arguments, as in ([7], pg. 243-256), and for which the Taylor series expansions hold true

$$\sin x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \tag{1}$$

$$\cos x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}. \tag{2}$$

Further, the exponential function has an important role as well, which is denoted by $\exp$ or e and for which it holds true that

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$$\exp(x)=\sum_{n=0}^{+\infty}\frac{x^n}{n!}. \tag{3}$$

Let's state that $\exp$ is monotony increasing function on $\mathbb{R}$ and that it looks completely different from the functions $\sin$ and $\cos$, and thus the students which study these functions through the real numbers do not expect any relations between the functions $\exp$, $\sin$ and $\cos$. Indeed, it seems that these functions are given by completely different sources, and that is the way they are introduced in secondary school. However, if in (3) we substitute the real variable x by the complex variable z we define the exponential function with complex variable:

$$\exp(z)=\sum_{n=0}^{+\infty}\frac{z^n}{n!},\quad z\in\mathbb{C}. \tag{4}$$

Further, using the Taylor series expansions (1), (2) and (4), wherein (4) we take that $z=ix$, $x\in\mathbb{R}$ and if we apply that

$$(ix)^{2k}=i^{2k}x^{2k}=(-1)^k x^{2k}$$

and

$$(ix)^{2k+1}=i^{2k+1}x^{2k+1}=i(-1)^k x^{2k+1}$$

we get the following

$$e^{ix}=\exp(ix)=\sum_{n=0}^{+\infty}\frac{(ix)^n}{n!}=\sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n}}{(2n)!}+i\sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!},$$

i.e.

$$e^{ix}=\cos x+i\sin x, \tag{5}$$

Thus, we obtain the Euler formula as a formal identity. Clearly, this is not the proof of the formula (5), but is methodically good enough explanation for its further applying.

2. INTRODUCING THE EULER FORMULAE IN A SECONDARY EDUCATION

In the introduction part we present how by using the Taylor series expansions of the functions $\exp$, $\sin$ and $\cos$ we can obtain the Euler formula as a formal identity. As already said, this procedure is inapplicable to secondary school students, so in this section we will give a procedure that has been deemed acceptable for secondary school students and that it is sufficient for students to "convince" the accuracy of the Euler formula, so it can still be used. For this purpose, initially we will prove the following theorem.

Theorem 1. *Let the function* $f:\mathbb{R}\to\mathbb{C}$ *be defined by*

$$f(\alpha)=\cos\alpha+i\sin\alpha,\quad \alpha\in\mathbb{R}.$$

Then,

a) $f(\alpha)\neq 0$, *for each* $\alpha\in\mathbb{R}$.

b) $f(\alpha+\beta)=f(\alpha)f(\beta)$, *for all* $\alpha,\beta\in\mathbb{R}$.

c) $f(-\alpha)=\frac{1}{f(\alpha)}$, *for each* $\alpha\in\mathbb{R}$.

Proof. a) Let there exist $\alpha\in\mathbb{R}$, so that $f(\alpha)=0$. Due to this, it exists $\alpha\in\mathbb{R}$, so that $\cos\alpha+i\sin\alpha=0$, that is $\cos\alpha=\sin\alpha=0$, which is contradictory to the basic trigonometric identity

$$\cos^2\alpha+\sin^2\alpha=1.$$

b) For all $\alpha,\beta\in\mathbb{R}$ it holds true that

$$\begin{aligned} f(\alpha+\beta)&=\cos(\alpha+\beta)+i\sin(\alpha+\beta)\\ &=\cos\alpha\cos\beta-\sin\alpha\sin\beta+i(\sin\alpha\cos\beta+\cos\alpha\sin\beta)\\ &=(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)\\ &=f(\alpha)f(\beta). \end{aligned}$$

c) For each $\alpha\in\mathbb{R}$ is true that

$$\begin{aligned} f(-\alpha)&=\cos(-\alpha)+i\sin(-\alpha)\\ &=\cos\alpha-i\sin\alpha\\ &=\frac{(\cos\alpha-i\sin\alpha)(\cos\alpha+i\sin\alpha)}{\cos\alpha+i\sin\alpha}\\ &=\frac{1}{\cos\alpha+i\sin\alpha}=\frac{1}{f(\alpha)}. \blacksquare \end{aligned}$$

Further, it is to notice that in the above theorem we proved that the function f satisfied the properties of exponential function, thus, it is natural to introduce the notation $f(x)=e^{ix}$, for each $x\in\mathbb{R}$, which practically means introduction of Euler formula (5). Then, the properties b) and c) of theorem 1 are expressed as following

$$e^{i\alpha}e^{i\beta}=e^{i(\alpha+\beta)} \tag{6}$$

$$e^{-i\alpha}=\frac{1}{e^{i\alpha}}, \tag{7}$$

and we announce to the students that the formulae (6) and (7) and the principle of mathematical induction directly imply the following

$$(e^{i\alpha})^n=e^{in\alpha},\ \text{for } n=0,\pm1,\pm2,\dots. \tag{8}$$

In the further considerations, we remind the students about the trigonometric entry of complex number and conclude that each complex number z, such that $|z|=1$ and $\varphi=\arg z$ can be expressed as

$$z=\cos\varphi+i\sin\varphi=e^{i\varphi}, \tag{9}$$

and it holds true that $e^{2\pi i}=1, e^{\pi i}=-1, e^{\frac{\pi i}{2}}=i, e^{\frac{3\pi i}{2}}=-i$. Finally, by applying the properties of trigonometric functions sin and cos and in (9) by substituting φ with $-\varphi$ the students have to be convinced to the accuracy of the following formula

$$\cos\varphi - i\sin\varphi = e^{-i\varphi}, \tag{10}$$

Using (9) and (10) we obtain the well known Euler formulae:

$$\cos\varphi = \frac{e^{i\varphi}+e^{-i\varphi}}{2}, \quad \sin\varphi = \frac{e^{i\varphi}-e^{-i\varphi}}{2}, \tag{11}$$

which allows us to express the trigonometric functions cos and sin in terms of exponential function.

After introducing the Euler formulae, it is desirable for the students to be assured of the strength of the adopted system, so it is recommended for them to solve several problems like the following.

Example 1. *Let* $\alpha, x \in \mathbb{R}, n \in \mathbb{N}$ *. Find the sums:*

a) $A = \cos x + \cos(x+\alpha) + \cos(x+2\alpha) + ... + \cos(x+n\alpha)$, and

b) $B = \sin x + \sin(x+\alpha) + \sin(x+2\alpha) + ... + \sin(x+n\alpha)$.

Example 2. *Let* $\alpha \in \mathbb{R}, n \in \mathbb{N}$ *. Prove the following:*

a) $1 + \cos\alpha + \cos 2\alpha + ... + \cos n\alpha = \dfrac{\sin\frac{(n+1)\alpha}{2}\cos\frac{n+\alpha}{2}}{\sin\frac{\alpha}{2}}$,

b) $\sin\alpha + \sin 2\alpha + ... + \sin n\alpha = \dfrac{\sin\frac{(n+1)\alpha}{2}\sin\frac{n+\alpha}{2}}{\sin\frac{\alpha}{2}}$.

3. EXPONENTIAL ENTRY OF COMPLEX NUMBER. GEOMETRICAL INTERPRETATION OF PRODUCT AND QUOTIENT OF COMPLEX NUMBERS

Once the Euler formulae are adopted, the students should recall the trigonometric entry of a complex number, i.e. the following entry

$$z = |z|(\cos\varphi + i\sin\varphi), \tag{12}$$

where $\varphi = \arg z$ is a main value of $\operatorname{Arg} z \in (0, 2\pi]$ and from (9) and (12) to conclude that each complex number $z \neq 0$ can be expressed as following

$$z = re^{i\varphi}, \tag{13}$$

where $r = |z|$ and $\varphi = \arg z$. After that, the students are told that the entry (7) of a complex number $z \neq 0$ is called exponential entry of z .

In the further consideration, it has to be noticed that using the formulae (6) and (7) and the exponential entry of complex number (13), the operations multiplication and division of complex numbers can be written as following:

$$z_1 z_2 = r_1 e^{i\varphi_1} r_2 e^{i\varphi_2} = r_1 r_2 e^{i(\varphi_1+\varphi_2)}, \tag{14}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\varphi_1}}{r_2 e^{i\varphi_2}} = \frac{r_1}{r_2} e^{i(\varphi_1-\varphi_2)} . \tag{15}$$

In this section, before we pass to the application in geometry, it is desirable to provide some additional comments. For example, it is good to be mentioned that if $z = re^{i\varphi}$, then (9) and (10) imply that $\overline{z} = re^{-i\varphi}$, thus if $\varphi = \arg z$, then $-\varphi = \arg \overline{z}$.

After adopting the exponential entry of complex number, it is necessary to pay attention on some advantages according to that, especially when the movements and similarities are studied. It can be achieved as following. Let E be a point with affix 1. Let's consider the points A and A' with affixes

$$a = \rho e^{i\theta} \text{ and } a' = \rho' e^{i\theta'},$$

respectively (figure 1). The product $b = aa'$ corresponds a point B, which is obtained as a third vertex of the triangle $OA'B$, if this triangle is constructed such that it is similar to the triangle OEA. Indeed, the similarity of these triangles implies that $\angle EOA = \angle A'OB$, that is $\arg b = \theta + \theta'$. For the same reasons the following equality holds true $\rho : 1 = |b| : \rho'$, i.e. it holds true that $b = \rho\rho'$, thus $b = aa'$.

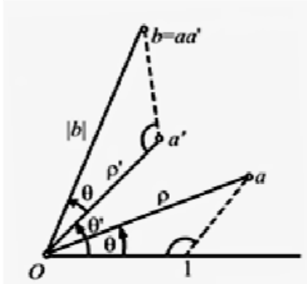

Figure 1

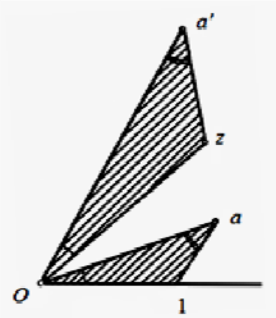

Figure 2

Further, the students should adopt that the point Z with affix $z = \frac{a'}{a}$ is obtained by the construction of a triangle OZA' which is similar to the triangle OEA. Indeed, the similarity of these triangles implies $az = a'$, thus $z = \frac{a'}{a}$, (see the figure 2).

After adopting the geometric interpretation of product and quotient of complex numbers, (as we noticed that the exponential entry of complex number is very useful tool) by using the formula $a^n = a^{n-1}a$ and the procedure for construction of affix of product and quotient of two complex numbers consequently can be constructed the points

$$..., A_{-3}, A_{-2}, A_{-1}, E, A_1, A_2, A_3, ... \tag{16}$$

whose affixes are the complex numbers

$$..., a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3, ..., \tag{17}$$

respectively. Namely, if $r > 1$ and $0 < \alpha < \pi$, then the students should find the points $A_2, A_3, ...$ (figure 3), whose affixes are complex numbers $a^2, a^3, ...$, by consecutive construction of similar triangles OEA_1, $OA_1A_2, OA_2A_3, ...$.

Further, the students on their own or with your assistance, have to notice that by applying this procedure but in opposite direction they actually construct the similar triangles

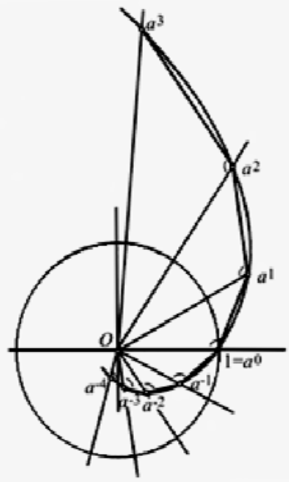

Figure 3

$$OEA_1, OA_{-1}E, OA_{-2}A_{-1}, OA_{-3}A_{-2}, \ldots$$

they find the points

$$A_{-1}, A_{-2}, A_{-3}, \ldots$$

whose affixes are the complex numbers

$$a^{-1}, a^{-2}, a^{-3}, \ldots .$$

Before considering the other possibilities for r and α, the students should be asked to consider the modules and the arguments of the powers. The objective of this consideration is the students to assume that modules of the powers increase or decrease by geometric and the arguments by arithmetic progression. During these considerations, and especially after the construction of the points (16), it can be seen that if we take that $\rho = r^n$, $\theta = n\alpha$ and eliminate n of these equations, we obtain that $\rho = r^{\frac{\theta}{\alpha}}$ and the students are told that all powers a^n are on the curve which in so called polar coordinates is given by the previous relation and it is known as logarithmic (Bernuolli's) spiral.

At the end of these considerations special attention should be paid to both cases first $r < 1$ and $0 < \alpha < \pi$, and second $r > 1$ and $-\pi < \alpha < 0$, whereby students cognitively should adopt that in these cases the logarithmic spiral is in the opposite direction of a spiral given on figure 3 and wrapping around the origin while θ grows. However, if $r < 1$ and $-\pi < \alpha < 0$, then the logarithmic spiral has the same form as shown on figure 3.

4. Conclusion

In secondary education algebraic and trigonometric entries of complex numbers, operation with complex numbers, De Moivre's formula and finding the square root of complex number are adopted. However, this knowledge is not quite enough for application of complex numbers in elementary geometry, and especially for studying the movements and similarities. For that purpose, it is necessary to adopt exponential entry of complex number, as well as geometric interpretation of multiplying and dividing complex numbers. The latter is possible only in higher levels of education. However, as shown in previous considerations, the students can adequately be "convinced" in the accuracy of the Euler formula and then to adopt the exponential entry of complex number. This procedure allows the following:

- students cognitively to acquire new knowledge and skills,
- prepare students for the study of the movements and the similarities with the appliance that allow complex numbers,

- improve the internal subject integration in math education,
- improve the preparedness of students for successful participation in higher levels of education.

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CENTERS OF HOMOTHETY OF CIRCLE CONFIGURATIONS

Sava Grozdev[1], Veselin Nenkov[2]

Abstract. Each triplet of circles in the plane defines 6 points, which are the homothetic centers of the circle couples. The paper examines geometric properties of the 6-point location. Further results are proposed for more than three circles.

1. INTRODUCTION

A natural idea in solving plane geometric problems with two or more circles is to use homothety. In many problems the identification of a homothety simplifies reasoning to high extent and leads to elegant geometric solutions. Some properties of the homothety are applied in the present paper together with combinations of circles to solve three problems for convex quadrilaterals from the papers of international mathematical competitions.

If three circles are given in the plane, then each couple of them generates a homothety. In such a way six points in the plane are defined, which are the centers of the corresponding homotheties. The geometric peculiarities of the six point locations in connection with configurations of three arbitrary circles ω_1, ω_2 and ω_3 are generalized in the following

Lemma 1. *The homotheties in the set of three circles ω_1, ω_2 and ω_3 have the properties:*

1) *The homothety h_{ij} with center S_{ij} of ω_i and ω_j can be represented as a composition of the homothety h_{ik} with center S_{ik} of ω_i and ω_k, and the homothety h_{kj} with center S_{kj} of ω_k and ω_j, S_{ij} lying on the line $S_{ik}S_{kj}$* $(1 \le i \ne j \ne k \le 3)$.
2) *The ex-centers of homothety for the three couples of circles are co-linear* [3], [4].
3) *Each couple of in-centers of homothety are co-linear with one ex-center of homothety.*
4) *The six centers of homothety are located in triplets on four lines* [4].

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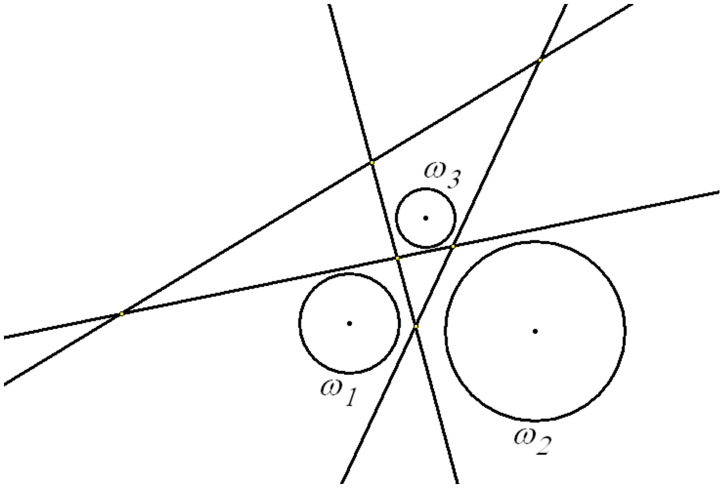

Figure 1

The assertions of lemma 1 are illustrated in Fig. 1. They express properties of the interaction between two circles through a third one.

By means of lemma 1 one can solve a problem, which contains a very complicated configuration of circles at first glance.

2. A CONFIGURATION OF FIVE CIRCLES IN A CONVEX QUADRILATERAL

Given is a convex quadrilateral K *, which is divided into 9 quadrilaterals by means of 4 lines as shown in Fig. 2. The lines meet the diagonals of the quadrilateral. It is known that circles can be inscribed in the quadrilaterals 1, 2, 3 and 5. Prove that it is possible to inscribe a circle in the quadrilateral 4 too.* (The problem is from the paper of the Zhautykov Olympiad in 2014.)

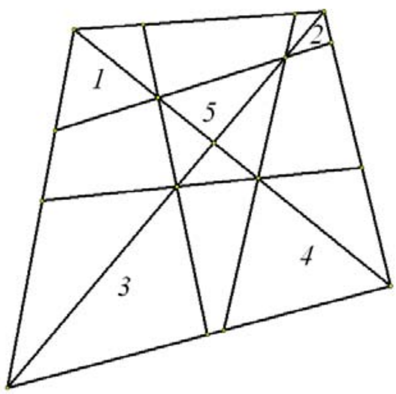

Figure 2

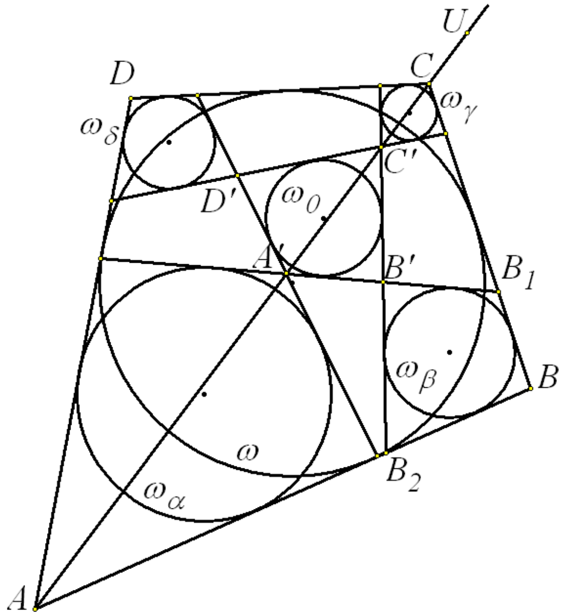

Figure 3

It turns out that in this configuration of circles one more circle is hidden and it is inscribed in the quadrilateral K (Fig. 3). The notations in Fig 3 will be used in the solution of the problem. The existence of the circle ω_β should be proved. The in-center of homothety of the circles ω_0 and ω_γ is the point C', while the in-center of homothety of the circles ω_0 and ω_α is the point A'. It follows from lemma 1, that the ex-center of homothety of the circles ω_γ and ω_α is the point U on the line $A'C' \equiv AC$. Let ω be the circle, which is tangent to the sides AB, BC and CD of the quadrilateral $K = ABCD$. The point C is the ex-center of homothety of the circles ω and ω_γ, while the ex-center of homothety of the circles ω_γ and ω_α is the point U on the line AC. According to lemma 1 this means that the ex-centers of homotyhety of the circles ω and ω_α is the point G on the line AC. On the other hand, the ex-center of homothety of ω and ω_α lies on their common outer tangent AB. Consequently, $G = AC \cap AB$, i.e. $G \equiv A$. Since the common outer tangents of two circles pass through their ex-center of homothety and AD is tangent to ω_α, then AD is tangent to ω through A too. Thus, ω is inscribed in $ABCD$. Now, let ω_β be the circle, which is tangent to BB_1, B_1B' and $B'B_2$. The point B' is the in-center of homothety of the circles ω_α and ω_0, while the in-center of homothety of the circles ω_δ and ω_0 is the point D'. According to lemma 1, this means, that the ex-center of homothety of ω_δ and

ω_β is the point V on the line $B'D' \equiv BD$. The point D is the ex-center of homothety of the circles ω and ω_δ, while the ex-center of homothety of the circles ω_δ and ω_β is the point V on the line BD. In such a way, it follows from lemma 1, that the ex-center of homothety of ω and ω_β is the point L on the line BD. On the other hand, the ex-center of homothety of ω and ω_β lies on their common outer tangent BC. Consequently, $L = BC \cap BD$, i.e. $L \equiv B$. As before, it follows from here that ω_β is tangent to AB, i.e. ω_β is inscribed in the quadrilateral $BB_1B'B_2$.

3. CO-LINEARITY OF THREE POINTS

The point P lies on the side AB of a convex quadrilateral $ABCD$. Let ω be the in-circle of the triangle CPD, and let I be its in-center. Suppose that ω is tangent to the in-circles of the triangles APD and BPC at the points K and L, respectively. Let the lines AC and BD meet at E, and let the lines AK and BL meet at F. Prove that the points E, I and F are collinear. (The problem is from the Short list of the 48-th International Mathematical Olympiad, 2007 in Vietnam.)

Except lemma 1, the solution of this problem uses the following

Lemma 2. *The quadrilateral $ABCD$ is circumscribed if and only if the in-circles of the triangles ABC and ADC are tangent.* [2]

Let Ω be the circle with center O, which is tangent to the segment AB and the rays AD and BC, while ω_α and ω_β are the in-circles of the triangles APD and BPC, respectively (Fig. 4). The point A is the ex-center of homothety of the circles Ω and ω_α, while K is the in-center of homothety of ω_α and ω. Consequently, the in-center of homothety U of Ω and ω lies on the line AK. On the other hand, B is the ex-center of homothety of Ω and ω_β, while L is the in-center of homothety of ω_β and ω. Therefore, the in-center of homothety U of Ω and ω lies also on the line BL. In such a way we get that $U \equiv F$. It follows from here, that the point F lies on the central line OI of the circles Ω and ω. Now note, that according to lemma 2, the quadrilaterals $APCD$ and $BPDC$ have in-circles. Denote them by Ω_α and Ω_β (Fig. 4). The ex-center of homothety of Ω and Ω_α is the vertex A of $ABCD$, while the ex-center of homothety of Ω_α and ω is the vertex C. Consequently, the ex-center of homothety V of Ω and ω lies on the diagonal AC. In the same way, passing through the circle Ω_β, we obtain, that V lies on the diagonal BD. Therefore, $V \equiv E$. Since E lies also on the central line OI, then E, I and F are co-linear.

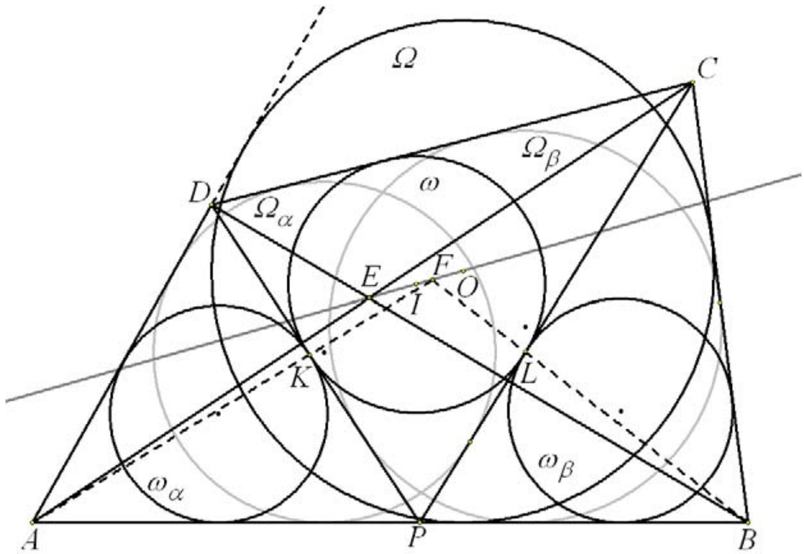

Figure 4

4. CO-LINEARITY OF THREE POINTS

Given is a convex quadrilateral $ABCD$, for which $|BA| \neq |BC|$. The in-circles of the triangles ABC and ADC are denoted by ω_1 and ω_2, respectively. It is known, that there exists a circle ω, which is tangent to the ray BA at a point after A, to the ray BC at a point after C and to the lines AD and CD. Prove that the common outer tangent to ω_1 and ω_2 meet on ω. (The problem is from the paper of the 49-th International Mathematical Olympiad, 2008 in Spain.)

Except lemma1 the solution of this problem uses also the following

Lemma 3. *If the quadrilateral $ABCD$ is circumscribed and the circumcircle is situated in $\sphericalangle ABC$, then*

1) *the ex-circle Ω_δ of ΔACD, which is tangent to the side AC, is tangent to the in-circle ω_1 of ΔACB too;*
2) *the ex-circle Ω_β of ΔACB, which is tangent to the side AC, is tangent to the in-circle ω_2 of ΔACD.* [1]

Let ω_1 and Ω_δ be tangent at the point U, while ω_2 and Ω_β be tangent at the point V (Fig. 5). The points U and V lie on the line AC and they are the ex-centers of the homotheties h_1 and h_2 between the corresponding couples of circles. The ex-center of the homothety h_{12} (the common point of the common outer tangents) of ω_1 and ω_2 is denoted by S .

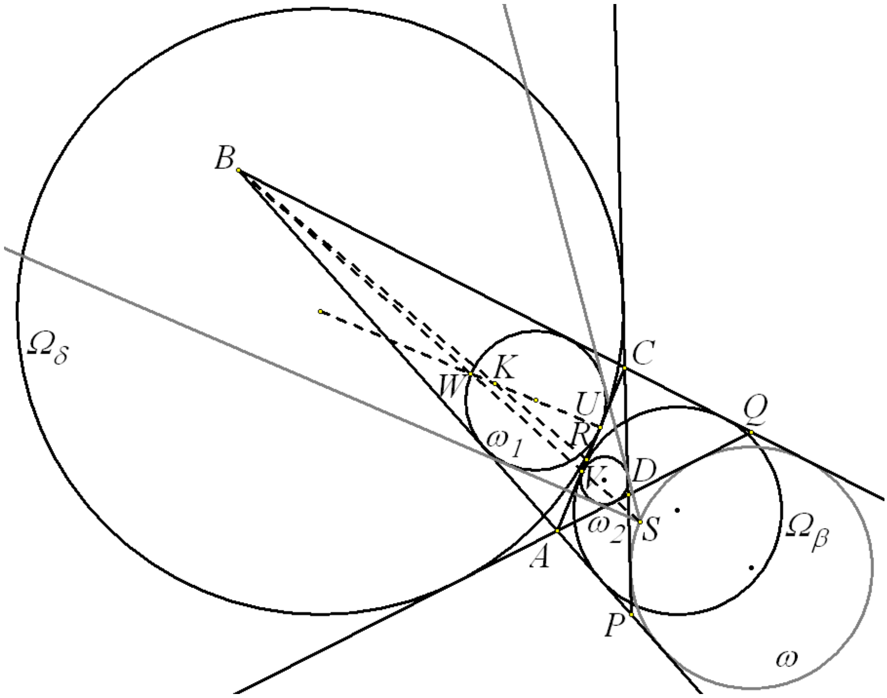

Figure 5

Note, that Ω_δ is transformed to ω_2 by the homothety h' with ex-center D , while Ω_β is transformed to ω_1 by the homothety h'' with ex-center B . It follows from lemma 1 that the homothety h_{12} can be represented as the composition of the homotheties h_1 and h', thus the points U , D and S are co-linear (Fig. 5). Analogously, h_{12} can be represented as the composition of the homotheties h_2 and h'' , thus the points V , B and S are co-linear. Consequently, the lines UD and VB meet at the point S (Fig. 5). Now, let $\overline{h}_1$ be the homothety with center D , which transforms Ω_δ to ω, while $\overline{h}_2$ be the homothety with center B , which transforms Ω_β to ω. More, let $\overline{h}$ be the homothety with in-center between Ω_δ and Ω_β. Then U is the image of V under $\overline{h}$. The homothety $\overline{h}$ can be represented is the composition of $\overline{h}_1$ and $\overline{h}_2^{-1}$. Consequently, the center of $\overline{h}$ lies on the line BD . On the other hand, this center lies on the common inner tangent of the circles Ω_δ and Ω_β . Thus, the center of $\overline{h}$ is the common point R of the inner diagonals. The homothety $\overline{h}_1$ transforms U to the first common point T of the line UD and ω, while the homothety $\overline{h}_2$ transforms V to the first common point T' of the line VB and ω. But, by means of the composition of the homotheties $\overline{h}$ and $\overline{h}_1$ the point V is transformed to T. Consequently, $T' \equiv T$. In such a way we have obtained, that the lines UD and VB meet at the point T on ω. As shown before, the same lines meet at the point S. Consequently, it follows, that the ex-center of homothety S of ω_1 and ω_2 is a point on ω. This ends the solution of the problem.

5. CONCLUSION

Some other curious facts, connected with this problem, can be found in [1].

As a conclusion we note, that in all the considered examples the application of a homothety is connected with the use of one or more hidden circles.

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INTELLECTUAL AND EMOTIONAL MATURITY OF STUDENTS IN SOLVING PROBLEMS OF MEASURING TIME

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Abstract. Younger students face with the concept of measuring time for the first time in the second grade. In this paper I will establish whether and to which extent they are able to understand and solve problems of measuring time. Abstract situations are analyzed, didactic materials are used and correlation with other school subjects is made.

1. INTELLIGENCE AND EMOTIONAL INTELLIGENCE

1.1. INTELLIGENCE

The definition of intelligence is derived from the Latin word ***inter,*** meaning "between", and ***legere,*** meaning "gather", "pick". The literal translation of the term intelligence would be recognition of interconnections and interrelationships between the terms we encounter and on which we base our assumptions about the world we live in.

The ever present dilemma when talking about intelligence is whether it represents the ability to solve new problems and deal with unfamiliar situations or the ability for abstract thinking. Is it the ability to understand causes and consequences of a particular problem or distinguishing between what is important and unimportant, or is it the ability of easy and quick learning and acquiring new skills?

It is generally agreed that there are three basic categories of intelligence:

1. *Abstract or verbal intelligence* refers to one's ability to use concepts and terms (words), to realize their meanings and use them properly.
2. *Practical intelligence* refers to dexterity in handling objects and things found in the environment we live in (manipulative abilities), as well as the ability of psychomotor reactions to problematic situations.
3. *Social intelligence* refers to the ability to interact with other people.

The following question was constantly imposing: what influences the development of intelligence?

Hereditary factors partly influence the development of intelligence. However, one should be careful here! They do not refer to the things we genetically inherited from our parents and ancestors, but to the things they provided for us intellectually speaking (in the intellectual sense).

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The other very important factor is our *environment*, especially during the developmental period when it has greater impact on our lives (school, society).

Both factors are extremely important for the development of intelligence, and they both act cumulatively.

The development of intelligence ends with the entry into adulthood 18-25, and later on it is only enriched by the amount of knowledge. However, it has been proven that in people who engage in intellectual work in their later years, intelligence does not deteriorate or stagnate, but stays at a high level.

1.2. EMOTIONAL INTELLIGENCE

Many psychologists, doctors and neuropsychiatrists have dealt with the term emotional intelligence. One of the most fundamental theories of emotional intelligence was developed by Daniel Goleman [5] in his book *Emotional Intelligence*. (Daniel Goleman*, Emotional Intelligence: Why It Can Matter More Than IQ*, Bantam Books, 1995). See also [2],[3],[7],[12],[14].

According to Daniel Goleman, the emotional intelligence model consists of several important components:

1. Self-awareness - one's ability to understand personal emotions and their impact on other people.
2. Decision making - the analysis of one's actions and concept of consequences.
3. Self-regulation - the knowledge of what is underneath particular feeling.
4. Stress management - learning how to relax and understand the importance of relaxation.
5. Empathy – understanding of other people's emotions and respecting other people's opinions.
6. Communication – talking about feelings, understanding them and being a good listener.
7. Self- revelation – understanding the necessity for openness and trust, learning how and when to speak about your emotions.
8. Observation – recognizing patterns in personal life and lives of others
9. Self –acceptance – accepting one's flaws, appreciating one's virtues.
10. Personal responsibility – taking responsibility and recognizing the consequences of your decisions and actions.
11. Self-confidence – the ability to express your concerns and feelings without anger and passivity.
12. Group dynamics – realizing when to lead and follow.
13. Resolving conflicts – "win/ conquer" model in negotiation, compromise etc.

Some scientists defined emotional intelligence as *the ability to monitor and distinguish their own emotions and feelings, as well as those of others* and use the information as a guide to opinions and behavior.

Revised emotional intelligence includes:

1) The ability of quick observation and emotional expression
2) The ability to grasp and generate feelings which facilitate thinking processes
3) The ability to control emotions in order to promote emotional and intellectual growth.

Emotional intelligence is defined as the ability to *observe, assimilate, understand and manage emotions*. The given abilities are arranged according to the complexity of psychological processes, starting from more simple (observing and expressing emotions) to more complex (consciousness, reflexivity and managing of emotions).

In other words, emotional intelligence is the ability to observe emotions, deal with them and provoke them so that they could help the process of thinking. Emotional intelligence thus refers to the ability to:

a) Effectively maintain the relationship between emotions and thoughts.
b) Use emotions in order to improve reasoning.
c) Reasonably analyze emotions.

2. CONTENT KNOWLEDGE

Content knowledge has always been a significantly important factor of teaching and the main source of knowledge acquisition, acquisition of skills and habits as a starting point and basis for the development of scientific view and personality in general. See, for example, [1],[4],[6],[9],[10],[13],[15].

Content knowledge, as well as educational plan and program represent a document which contains educational aims of a particular subject and which every teacher is required to have. The program indicates how those aims should be achieved and provides teachers with basic methodological instructions and different ways of teaching.

In order to successfully implement content knowledge, a teacher has to:

1. Be fully familiar with the program of the subject he teaches so that he could implement it;
2. Be familiar with the programs of other subjects in order to be able to successfully make correlation between them.
3. In order to realize the program requirements, a teacher has to pay attention to teaching rhythm and pace.

The selection of content knowledge is defined by the official documents that greatly determine educational (general, technical, professional) and pedagogic basis of teaching.

The Ministry of Education, Science and Technological Development of Republic of Serbia determines content knowledge, educational aims and tasks by its regulations and laws, and in the past few years, learning standards and students' achievements as well.

According to the traditional understanding of teaching, three main factors make the famous didactic triangle: *student- teacher- content*.

2.1. The Analysis of math operational plan for the second grade primary school students

Educational aims and objectives for the second grade math teaching, unit *Measurement and measures*, are the following:

- Define the concept of time as a measurement.
- Establish the concept of time units (hour, minute, day, week, and month).
- Acquire understanding of the relationships between time units (hour, minute, day, week, and month).
- Teach students to use time units in everyday life.
- Teach students how to convert one time unit into another
- Teach students to tell time and use a calendar
- Teach students how to perceive time – time units.

During the school year, four classes are dedicated to acquiring these goals.

Topic	Number	Theme Unit	Type of class	Approach	Method	T. aid	Standards for assessing Student Achieve.
Measurements/ measuring	176.	Year, month, week, day	NT	F,I	D, WP, GA	calendar	1MA. 2.4.2.
	177.	Year, month, week, day	R	F,I	D, WP		1MA. 2.4.2.
	178.	Hour, minute	NT	F,I	D, WP DM, GA	clock	1MA. 2.4.2.
	179.	Hour, minute	R	F,I	D, WP GM	A clock	1MA. 2.4.2.
	180.	We have learned this in the second grade	Sys	F,I	D, WP		

2.1.1. Define the term – type of class

* Type of class: NT – introducing a new topic; R – revision; Sys – systematization.

Introducing a new topic in class is central for the teaching process. During this type of class basic educational goals are achieved – knowledge acquisition, and to a certain degree development of students' skills.

In the teaching process most of the classes are dedicated to introducing the new topic. One of the main reasons for this is that *teaching programs are too detailed and demanding, as well as the fact that teachers are not trained enough to deal with this problem by applying more adequate methods and by a better organization of lessons.* Teachers are racing to achieve all the objectives of a program and introduce all the teaching units. Unfortunately, this aspect of teaching is mostly monitored and evaluated.

Revision class is dedicated to production of newly gained knowledge and making it a part of student' skills and habits. During revision classes students complete functional tasks, some students can *improve their knowledge*, and those who have *some doubts or uncertainties* can clarify them.

In these classes students have the chance to check their knowledge, become more secure in using newly gained knowledge and skills, and are motivated to further improve their knowledge.

Systematization is done at the end of a particular unit, period or term and at the end of the school year.

The aim of systematization is to remind students what they have learned and done in the past period of time, and the knowledge they have gained. It should not be strange if students only recognize some parts of lessons because they are prone to forget a lot of received data.

2.1.2. Clarification of the term teaching approach

** Teaching approach: F – frontal teaching; I – individual work.

Frontal teaching represents cooperative work of all students under the guidance of a teacher. The teacher designs the lessons and guides students through them, and then collects student responses. All the students complete the same tasks and solve the same problems and are focused on the teacher. During the learning process students usually do not establish relationships with one another. Learning is considered to be an individual task of each student, either as a process of receiving information from the teacher, or a process in which students are *required to give answers to the teacher's questions.*

In this type of teaching, students work under the supervision of their teacher at the same time, on the same problem and goal. The teacher observes the class as whole, and not each student individually. Frontal teaching is the most effective form of teaching with respect to time a teacher invests and an effort he makes.

Individual work requires of each student to complete a particular assignment, *alone, as an individual,* without the possibility of exchanging information with other students. Students complete the assignments given by their teacher by themselves. Assignments can be given according to the abilities of each student, and according to didactic goals.

Individual work can be used for revision, evaluation and improvement of knowledge. Moreover, it is adapted to the abilities *and tempo of learning* of individual students. The benefits of this type of learning are great because students gain certain knowledge *at their own pace and intellectual effort.*

2.1.3. Clarification of the term *teaching method*

*** Teaching methods: M – monologue method; D – dialogue method; DM – demonstrative method; WP– written papers; GA– game activities

Monologue method is a type of verbal teaching where the teacher himself introduces a new topic; it is also known as the oral presentation method. It is characterized by *the active role of a teacher and the passive role of students.* The students may seem to be inactive and just listening to the teacher on the surface. Judging by this some may draw the conclusion that this type of activity is reproductive in its nature, which is wrong. Students follow the presentation only if the teacher succeeds to motivate them and direct their attention to the achievement of certain goals and assignments.

Pedagogic significance of this method is in its possibility to unambiguously present target knowledge. It can be presented systematically, logically and for a short period of time.

Dialogue method is a method in which the teacher and students work together in order to achieve a certain goal. *It is based on a series of questions and answers,* the dialogue between the teacher and students, is one of the main characteristics of this method.

The dialogue enables the teacher to talk with students about different topics, exchange thoughts, express ideas that is why it is *contemplative communication*. The dialogue enables the teacher to provoke contemplative activity of students, and at the same time, based on their answers check whether *the questions are appropriate and his actions justified*. Thus, the teacher can, if it is necessary, *make some changes with respect to the scope or depth of the knowledge he intends to provide.*

Demonstrative method is a method of demonstration. As a teaching method it involves complex didactic activities of teachers, such as: *displaying or showing pictures of different objects, demonstrating certain phenomena, events and processes.*

There are strong arguments against learning based on *passive observation,* and acceptance of materials displayed by the teacher. It is justifiably pointed out that it is not enough to show pictures to students or draw their attention to certain details, and get their impressions as a basis for further concepts and abstract principles (See, Hans Ebli, *Psychological Didactic*). The teacher is required to adequately stimulate contemplative activities of students so that they could compare what is being presented to them. In other words, the basic element of thinking is not what is being presented to students, but the *scheme of activities in which each student actively participates.*

Written papers are used more and more often in various forms and with the aid of different materials and instruments. In most cases, *written papers and tests on certain topics or with a certain number of questions* (written form of questions) are used regularly. These assignments can be given at school, for the whole class, a group of students or each student individually.

Different levels of knowledge and especially student's ability to work by himself can be assessed this way.

Written assignments and tests can be made by the teacher or found in teacher's books, books with assignments, objective tasks which teachers often use.

Game activities represent the best way of learning, revision, practice and assessment. Students achieve learning objectives *through games* in a relaxing end noisy environment, not realizing that they are actually learning. Students spontaneously show their creativity, individuality and resourcefulness.

Students adore classes in which game activities are used for revision and systematization.

2.1.4. Clarification of the term *teaching aids*

**** Teaching *aids* are didactically shaped, various technical and other material aids, manufactured or handmade, *adjusted to the process of learning and psychological and physical traits of students.* The aim of teaching aids is to bring reality closer to students, making it accessible and obvious. The use of teaching aids contributes to better understanding of lessons, greater involvement of students, and effectiveness at work in general.

In some cases, students are required to make a particular teaching aid by themselves. In that way, apart from creativity, students concurrently improve manual skills and thinking process.

***** Learning *standards and student achievements* are defined by The Regulations on standards that are available on the official website of The Ministry of Education, Science and Technilogical Development of Republic of Serbia.

2.2. CORRELATION WITH THE OPERATIONAL PLAN FOR THE SUBJECT WAU FOR THE SECOND-GRADE PRIMARY SCHOOL STUDENTS

Theme units for the school subject The World Around Us, their agenda (in November and the beginning of December), and its correlation with other school subjects (Math and Art).

Teaching m.	Number	Theme unit	Type of class	Approach	Method	Teaching aids	Learning standards
Measur. time	23.	Telling time	NT	F	D	A clock	1PD. 1.4.4.
	24.	The use of clock	R	Group work	D, X	A clock	1PD. 1.4.4.
	25.	A week, month and year	NT	F	Illustr. method	A calendar	1PD. 1.4.5.
	26.	Seasons	NT	F	Illustr. method	A calendar	1PD. 1.4.5.
	27.	Revision	R	Group work	Text	Activity book	1PD. 1.4.5.
	28.	Interesting facts	R	I	X, P	Encyclopedia	1PD. 1.4.5.
	29.	Assessment	E	I	WP	Test	1PD. 1.4.5.

***** *Heuristic method,* also known as analytic method, directs students to look for solutions to the problems by themselves, especially by using inductive reasoning. It is used with the aim of gaining new knowledge.

The teacher gives an assignment or poses a problem, and *by means of carefully chosen questions he encourages students to use the knowledge they already have* in order to discover the truth, new schemes, to find solutions to problems and reach conclusions. Students are encouraged to solve problems and their search can be carried out with the aid of teaching aids or technical recourses.

Heuristic method enables teachers to guide their students while they are searching for answers, give them directions and help them gradually come to the right solution. The greatest advantage of this method is that it *enables students to learn by themselves, encourages them to be more involved in the learning process, increases their self-conscience, helps them become aware of their knowledge and understand what they have learned.*

There is *vertical and horizontal correlation between* the subjects thought in the same or different grades, and theme units overlap. The complexity of content knowledge is tempting and it would be a pity not to take advantage of the subjects that are interconnected.

Horizontal correlation of content knowledge is a link between theme units and lessons of different school subjects taught in the same grade.

In this paper I established the correlation between Math, The World Around Us and Art. Using different teaching methods, I managed to explore the theme *Measuring Time* with my students, a theme abstract for their age.

3. RESEARCH AND DATA ANALYSIS

3.1. MAIN METHODS AND PROCEDURES OF THE STUDY OF PEDAGOGICAL ISSUES

Different methods, procedures and instruments can be used in pedagogic research. Various factors may influence their classification; however, they are divided into three basic types:

- *Fundamental research* which is mainly theoretical, directed towards basic questions in pedagogy such as: nature and character of education, learning in the teaching process, teaching principles, the process of shaping students into becoming moral, etc.
- *Developmental research,* also known as futuristic or prognostic, is directed towards the study of the ways in which education is being developed under special conditions.
- *Applied research* is practical, empirical, operative, directed towards the change and improvement of education, as well as the search for the most efficient ways of implementing knowledge found by means of fundamental research (for example, the use of new teaching method, or content knowledge, course books or teaching strategies.

Observing *research method* as a process which enables scientific study and leads us to a concrete pedagogic issue, or in other words a number of issues that can be examined by means of different procedures and instruments, I used the following research methods:

- *Method of theoretical analysis,* which performs critical analysis and questions certain attitudes, concepts and systems,
- *Comparative method* which compares one pedagogical phenomenon with the other makes comparison between results, values and achievements of the past and present in relation to certain teaching processes, themes and scope.
- *Experimental method* which is directed towards the study of cause and effect relationships between phenomena.
- *Descriptive method* and systematic non-experimental studies, often used for monitoring and assessment of student's progress.

Various research procedures, also referred to as *techniques or instruments,* are used within research methods**.** They are primarily intended for gathering facts needed for research.

One of the most important technique is *people observation,* which implies observation of their behavior, actions, reactions to certain situations and events. Observation can be performed individually, in a group, directly and indirectly, for a short period of time, for a long period of time, continuously... However, each observation which has scientific and research goal has *to be well planned, organized and systematic.*

3.2. TESTING - RESEARCH PROCESS; TEST – INSTRUMENT

Testing is a research process often used in pedagogic research. The instrument used for testing is a test. Test consists of a series of logically connected assignments; they are related to a certain theme unit and aimed at defining or measuring something, or in other words expressing a certain measurement.

Knowledge assessment test is used for establishing the quality and quantity of knowledge, as well as the extent to which students are able to use what they have learned from one or more school subjects

During the assessment class students got already prepared and typed tests (since these are younger students) with the assignments similar to those done in the past or found in textbook.

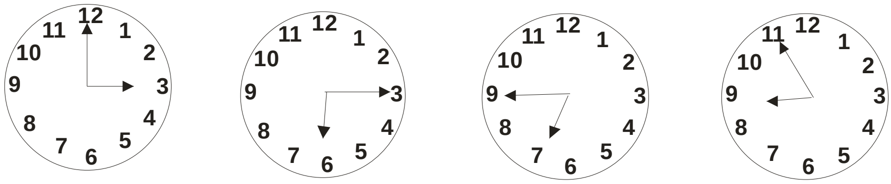
The Test:

1. Write down months of the year and the number of days of each month when the year is not a leap year.

1. January ______	5. ________________	9. _________________
2. February _____	6. ________________	10. ________________
3. _____________	7. ________________	11. ________________
4. _____________	8. ________________	12. ________________

2. A baby was born on April 1. It is 3 months and 5 days old. How many days has it been since the baby was born?

3. What time is it? Tell the time in the morning and afternoon.



4. If you go to bed at 9 p.m. and you wake up at 6:30 a.m., how long have you been sleeping?

5. Ana got on the bus that departed from Niš at 11:50 and headed to Belgrade. The bus trip took 2 hours and 15 minutes. When did Ana arrive to Belgrade, if we know that the bus was delayed for 20 minutes?

* One gets a certain number of points after completing each assignment
First assignment 20 points; Fourth assignment 15 points;
Second assignment 15 points; Fifth assignment 30 points;
Third assignment 20 points;
Maximum score on this test is 100 points or SCORES.

Frequency represents the frequency of occurrence of the same results in one group. The sample comprised 143 second-grade students. *Frequency of scores* 55 = 143.

Number of subjects, denoted by the letter N, represents the total number of participants of a study (sample size). This sample comprised 143 students, N=143.

Limitations of scoring system need to be established, since a series of scores does not indicate anything, apart from having all the scores of particular testing. The recorded scores have to be arranged from the highest to the lowest.

In this research I have encountered one problem. Two of 143 tested students were children with special needs. Their score was 0. In order to obtain concrete results, I did not take their tests into consideration. Therefore, I analyzed the obtained data from 141 students, score boundaries being: 30 the lowest, 85 the highest.

The range of scores – the range of a set of data represents the difference between the highest and lowest score. In this research it is number 55 (85—30).

Interval – scores are arranged in a series, they are further divided into groups, and each group contains a certain number of intervals. Since the upper quartile is 55, I intend to use groups with interval 5. The scores are shown in Table 1.

Number	Interval	Frequency X	Groups with the interval 5	The sum of all X
1.	81-85	15	8x85 3x84 3x83 1x81	1262
2.	76-80	9	2x80 2x79 3x78 2x77	706
3.	71-75	7	1x75 1x74 3x73 2x71	510
4.	66-70	3	2x69 1x67	205
5.	61-65	10	1x65 1x63 3x62 5x61	619
6.	56-60	36	8x60 9x59 6x58 5x57 8x56	2092
7.	51-55	24	7x55 6x54 6x53 2x52 3x51	1284
8.	46-50	14	5x50 4x49 2x48 3x41	581
9.	41-45	9	2x44 2x43 2x42 3x41	381
10.	36-40	8	2x40 1x39 5x36	299
11.	30-35	6	2x35 1x34	194

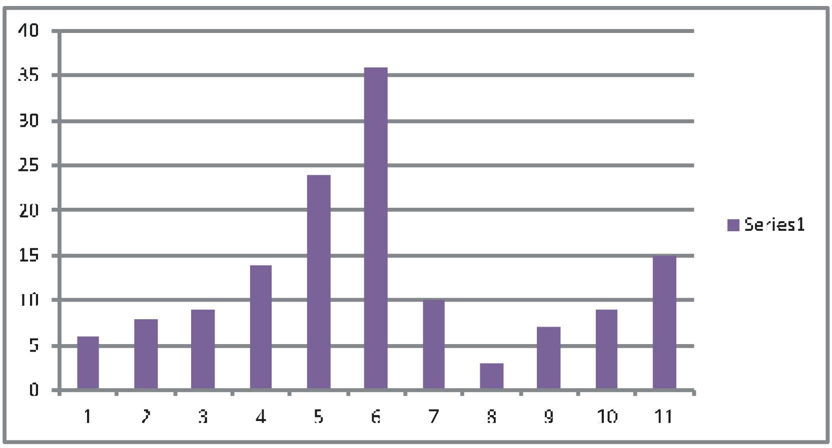
Average values are values which determine the average result of a series of scores. They are also known as measures of central tendency. The most important among them are arithmetic mean and median.

Arithmetic mean also called the average, denoted by the letter M is obtained when the number of scores is divided by the number of cases (There is not an established way of denoting certain values in pedagogy; therefore arithmetic mean can be denoted by AM, average values by AV, of X. In this research M=56. 97163.

Median is a value which divides the series of scores on the frequency of distribution scale into two halves, so that 50% of cases are below and 50% above that value.

In this research the median, denoted by Md is, Md=57.5.

Other statistical values such as: quartile and decile can be calculated this way.



Graphicon 1.

Quartile can be the first or the third. *The first quartile,* denoted by Q_1 means that 25% of subjects in the data set lie below Q_1 in this research *their score is* 51. *The third* *quartile,* denoted by Q_3 means that 75% of subjects in the data set lie below Q_3, in this research *the score is* 62.

The first decile means that 10% of cases lie below and 90% above that value. In this research *its score is* 36.

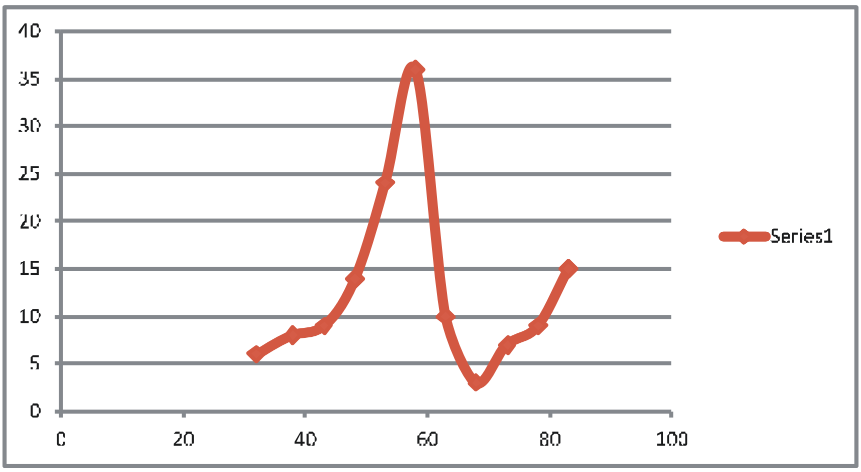
All these measures serve for *establishing scattering, scheduling and distribution of scores in a series.*

Graphical representation of results is used to figuratively and clearly show the results. By looking at the graphicon it can be concluded how the scores are grouped, where they are thick, where compact, the number of extreme cases of average values, etc.

The scores have to be ordered in order to be graphically displayed, see Table 1.

On a histogram, frequencies are displayed vertically, and group *limitations horizontally,* we began with the highest category. Graphicon 1.

It is the same case with the *frequency polygon*, apart from the fact that *average values* are displayed horizontally (they are calculated by dividing the sum of higher and lower limitation by 2) and score frequencies, vertically.



Graphicon 2.

However, average values do not provide information about the nature of a series of scores during one testing.

In order to find that out, *measures of variability* have to be established.

Standard deviation, denoted by *SD*, is the most important. It is used as a basic method of statistical research and in different scientific fields.

If standard deviation was ideal, all the scores above and below would equally deviate from it, and the polygon would be *bell shaped*.

This type of curve is called normal curve or Gaussian curve.

Based on this curve, below which each field has its surface and meaning, and each point its sigma distance, the true value and meaning of standard deviation is established.

The curve one gets during a particular research can never have a shape as the normal curve (its arithmetic mean and median do not match). However, it is used for the clarification of score distribution. The standard deviation and normal curve are a base for the implementation of statistical methods in pedagogic research.

Sample mean is a value which we may expect when conducting an experiment. It is estimated when the sum of scores is divided by the number of cases. It is denoted by X_n, where n stands for the number of cases. In this research, the sample mean is $X_n = 58,2057$.

Sample dispersion represents the mean square distance from the sample mean to the realized value. It is denoted by S_n^2, and is calculated by means of the following formula: $S_n^2 = \frac{1}{n}\sum X_k^2 - (EX)^2$. In this research, the sample dispersion is $S_n^2 = 187,72$.

Sample standard deviation represents the mean distance from the sample mean to the realized value. It is denoted by $\sqrt{S_n^2}$ and calculated as the root of dispersion. In this research, the sample standard deviation is $\sqrt{S_n^2} = 13,701$.

Corrected dispersion represents a corrected grade of mean square distance of sample mean of realized value in comparison with the dispersion. It is denoted by $\widehat{S_n^2}$, and calculated by means of the following formula: $\widehat{S_n^2} = \frac{n}{n-1} S_n^2$. In this research $\widehat{S_n^2} = 189,061$.

Corrected sample standard deviation represents corrected mean distance of sample mean of the realized value in comparison with the sample standard deviation. It is denoted by $\sqrt{\widehat{S_n^2}}$, in this research $\sqrt{\widehat{S_n^2}} = 13,75$.

4. CONCLUSION

Based on this research, conducted on the second grade primary school students who encountered the concept of measuring time for the first time, we can draw the following conclusions:

- *That during the process of learning new terms,* the terms are easier formulated if we use materials that can be perceived by our senses (in this case it was a didactic material – a clock model). Analyzing, clarifying and combining terms, enables the connections and relations to be simplified. In school conditions there is not enough time for students to discover new terms and processes on their own based on those they already know, therefore; we are forced to draw their attention directly to shared characteristics and acquaint them with concrete cases. This is the only way to formulate all the important concepts quickly (considering time limitations).
- *That emotional intelligence* which represents the ability to deal with emotions in order to improve emotional and intellectual growth, and at the same time

maintain the relationship between emotions and thinking, is seriously jeopardized in this research, causing as a consequence frustration, dissatisfaction, fear (of punishment), bad temper and insecurity in students.

- *That correlation* between different school subjects does not exist, there is even a time distance between them during a school year, even though content knowledge is connected and intertwined. Unless the teacher is wise and capable enough to connect the content knowledge (consolidate lessons and units), for which he needs to know the programs of subjects he teaches in detail, he usually cannot make his students realize the connections between the teaching units. The fact that a lot of teachers "blindly" follow the suggested teaching plan and program in fear of making mistakes and omission is devastating and usually results in the loss of the thread that connects school subjects;
- *That math educational plan and program* for the second grade primary school students written by the Ministry of Education, Science and Technological Development of Republic of Serbia did not plan enough number of classes for this topic (only five, having in mind that the fifth is planned for systematization). As a consequence, teachers are left to quickly go through these lessons, there is not enough time for the clarification of certain terms and processes, not enough time for practice and production, and teachers have to follow the plan and program! For all the above given reasons, this unit is not understood and acquired as expected, as a consequence students, as well as teachers are frustrated, insecure and dissatisfied;
- *That these types of assignment* require high level of knowledge, concentration, abstract and logical thinking. Having in mind that the fifth assignment was not completely solved by any of the second grade students in this research, we conclude that certain assignments on measuring time are inappropriate for children at this age, therefore; this teaching unit should:
- *Be postponed until the students are older* and (or)
- *Have the increased number of classes* (especially revision classes), in order to better acquire this type of knowledge.

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