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Contents

PREFACE

No research of people could be named true science if it is not supported by a mathematical proof.
Reliability of the assertions in different subjects is problematic when application of any mathematical domain is missing, i. e. when there is no connection with mathematics.

Leonardo da Vinci

The present book is devoted to students of the last school grades, university students, teachers, lecturers and all lovers of mathematics who want to enrich their knowledge and skills in complex numbers and their numerous applications in Euclidean Geometry. Few countries in the world include complex numbers in their secondary school curriculum but even if included the volume of the corresponding content is quite insuffcient consisting of elementary operations and geometric representation at most. The signifcance of the complex numbers is far from a real recognition in known textbooks and scholar literature. The applications not only in mathematics but also in many other subjects are considerable and the present book is a strong proof of such a statement.

Mainly, the book will be useful for outstanding students with high potentialities in mathematics preparing themselves for successful participation in mathematical competitions and Olympiads. Other target groups are not excluded, namely those, whose representatives like to meet real challenges, connected with unexpected circumstances in problem solving.

The material in the book is divided into four chapters. The first one contains basic properties of the complex numbers, their algebraic notation, the notion of a conjugate complex number, geometric, trigonometric and exponential presentations, also interesting facts in connection with Reimann interpretation and the set Cn. The second chapter includes various transformations of complex numbers in the Euclidean plane like similarity, homothety, inversion and Mőbius transformation. The third chapter is dedicated to the geometry of circle and triangle on the base of complex numbers. Numerous theorems are proposed, namely: Menelau's theorem, Pascal's and Desargue's theorem, Ceva's and Van Aubel's theorem, Stewart's theorem, Ptolemy's theorem and others. Exercises and problems are included in the Fourth chapter: 122 examples with solutions and 160 solved problems pare proposed. Together with all the 138 theorems, lemmas and corollaries accompanied by 64 examples and 88 figures, the book turns out to be a rather exhaustive collection of the complex number applications in Euclidean Geometry.

A high just appraisal of the book is due to the numerous non-standard problems in it taken from the National Olympiads of Bulgaria, China, Iran, Japan, Korea, Poland, Romania, Russia, Serbia, Turkey, Ukraine and others but also from Several International and Balkan Mathematical Olympiads.

The authors express their sincere thanks to the Editorial House "Archimedes 2" for the decision to accept the manuscript and to support the appearance of the present book. Also, sincere thanks to the reviewers Assoc. Prof. Dr. Lidia Ilievska Gorachinova and Assoc. Prof. Dr. Veselin Nenkov for their helpful criticism, removal of mistakes and well-wishing advices, which contributed to the final quality of the book. Of course, different lapses are possible and we will be grateful to the readers in case they notice such and bring them to the attention of the Editor.

February, 2017
Skopje The authors

CHAPTER I
COMPLEX NUMBERS

1. THE CONCEPT OF COMPLEX NUMBER, BASIC PROPERTIES

1.1. Definition. *Complex number* $z=(a,b)$ is the ordered pair of real numbers a and b.

The set of complex numbers is denoted by $\mathbf{C}$, i.e. $\mathbf{C}=\{(a,b)\mid a,b\in\mathbf{R}\}$.

The complex numbers (0,0) and (1,0) are denoted by n and e, respectively.

The definition of a complex number directly implies that two complex numbers $z_1=(a_1,b_1)$ and $z_2=(a_2,b_2)$ are equal if $a_1=a_2$ and $b_1=b_2$

1.2. Definition. *Sum* of two complex numbers $z_1=(a_1,b_1)$ and $z_2=(a_2,b_2)$ is the complex number

$$z_1+z_2=(a_1+a_2,b_1+b_2).$$

1.3. Definition. *Product* of two complex numbers $z_1=(a_1,b_1)$ and $z_2=(a_2,b_2)$ is the complex number

$$z_1\cdot z_2=(a_1a_2-b_1b_2,a_1b_2+a_2b_1)$$

1.4. Theorem. The addition and multiplication of complex numbers, satisfy the already known laws of arithmetic. Namely:

i) $z_1+z_2=z_2+z_1$, *commutative property of addition,*

ii) $(z_1+z_2)+z_3=z_1+(z_2+z_3)$, *associative property of addition,*

iii) $z_1z_2=z_2z_1$ *commutative property of multiplication,*

iv) $(z_1z_2)z_3=z_1(z_2z_3)$, *associative property of multiplication*, and

v) $(z_1+z_2)z_3=z_1z_3+z_2z_3$, *distributive property.*

hold true for all complex numbers z_1,z_2,z_3.

Proof. *i)* Let $z_1=(a_1,b_1)$ and $z_2=(a_2,b_2)$ be any complex numbers. Thus,

$$(a_1,b_1)+(a_2,b_2)=(a_1+a_2,b_1+b_2)=(a_2+a_1,b_2+b_1)=(a_2,b_2)+(a_1,b_1)$$

i.e., $z_1+z_2=z_2+z_1$.

The properties *ii), iii), iv)* and *v)* can be proven analogously. ■

Let's state that when proving Theorem 1.4 we explicitly used (by coordinates) the commutative, associative and distributive properties of addition and multiplication of real numbers.

1.5. Theorem. Any complex number z satisfies the following equalities: $z+n=z$, $z\cdot n=n$ and $z\cdot e=z.$, where n denotes the additive identity, and e denotes the multiplicative identity.

Proof. Indeed, if $z=(a,b)$ is an arbitrary comblex number, then
$z+n=(a,b)+(0,0)=(a+0,b+0)=(a,b)=z$,
$z\cdot n=(a,b)\cdot(0,0)=(a\cdot 0-b\cdot 0,a\cdot 0+b\cdot 0)=(0,0)=n$, and
$z\cdot e=(a,b)\cdot(1,0)=(a\cdot 1-b\cdot 0,a\cdot 0+1\cdot b)=(a,b)=z.$ ■

1.6. Theorem. If $z_1+z_3=z_2+z_3$, then $z_1=z_2$.

Proof. Let $z_1=(a_1,b_1)$, $z_2=(a_2,b_2)$ and $z_3=(a_3,b_3)$ be complex numbers. Then,

$$z_1+z_3=(a_1,b_1)+(a_3,b_3)=(a_1+a_3,b_1+b_3) \text{ and}$$
$$z_2+z_3=(a_2,b_2)+(a_3,b_3)=(a_2+a_3,b_2+b_3).$$

Since the given equality $z_1+z_2=z_2+z_3$ and the Definition 1.1 we get the following

$$a_1+a_3=a_2+a_3 \text{ and } b_1+b_3=b_2+b_3.$$

Furthermore, the properties of real numbersimply that $a_1=a_2$ and $b_1=b_2$, and since Definition 1.1 $z_1=z_2$. ■

1.7. Theorem. For each complex number z there exists one and only one complex number w, so that $z+w=n$.

Proof. Let $z=(a,b)$ be an arbitrary complex number, and w be defined as $w=(-a,-b)$. We get,

$$z+w=(a,b)+(-a,-b)=(a+(-a),b+(-b))=(0,0)=n.$$

So, we proved the existence of a complex number w. The uniqueness is directly implied by Theorem 1.6. ■

In our further consideration, the complex number w, so that $z+w=n$, will be denoted by $w=-z$, and w is called to bean opposite complex number of z.

Let z and w be arbitrary complex numbers. The complex number $z+(-w)$ is called to be a substraction of the numbers z and w, and is denoted by $z-w$.

1.8. Theorem. For all complex numbers z_1 and z_2,the equality

$$(-z_1)\cdot z_2 = z_1\cdot(-z_2) = -(z_1z_2) = (-e)\cdot(z_1z_2),$$

holds true and there is no ambiguity in the notation $-z_1z_2$.

Proof. Let $z_1=(a_1,b_1)$ and $z_2=(a_2,b_2)$ be arbitrary complex numbers So,

$$(-z_1)\cdot z_2 = (-a_1,-b_1)\cdot(a_2,b_2) = (-a_1a_2+b_1b_2, -a_1b_2-b_1a_2)$$
$$= -(a_1a_2-b_1b_2, a_1b_2+b_1a_2) = -(z_1z_2).$$

But, the commutativelaw of multiplication holds true, therefore the above equality implies that

$$-(z_1z_2) = -(z_2z_1) = (-z_2)z_1 = z_1(-z_2).$$

Finally, since the Theorem 1.5 and the already proved equalities we get that

$$(-e)(z_1z_2) = -(e(z_1z_2)) = -(z_1z_2) = (-z_1)z_2 = z_1(-z_2). \blacksquare$$

1.9. Definition. *The absolute value of the complex number* $z=(a,b)$ is defined by

$$|z| = \sqrt{a^2+b^2}.$$

Thus, the absolute value of each complex number z is a non-negative real number.

1.10. Theorem. a) If $z\neq n$, then $|z|>0$ and $|n|=0$.

b) $|z_1\cdot z| = |z_1|\cdot|z|$, for all complex numbers z and z_1.

Proof. Let $z=(a,b)$ and $z_1=(a_1,b_1)$ be any complex numbers

a) It is obvious that

$$|n| = \sqrt{0^2+0^2} = 0.$$

If $z\neq n$, then $a\neq 0$ or $b\neq 0$, i.e. $a^2>0$ or $b^2>0$. Thus,

$$|z|^2 = a^2+b^2 > 0.$$

b) Since,

$$|z_1z|^2 = |(a_1a-b_1b, a_1b+b_1a)|^2 = (a_1a-b_1b)^2+(a_1b+b_1a)^2$$
$$= a_1^2a^2+b_1^2b^2+a_1^2b^2+b_1^2a^2 = (a_1^2+b_1^2)(a^2+b^2) = |z_1|^2|z|^2,$$

we get that $|z_1\cdot z| = |z_1|\cdot|z|$. $\blacksquare$

1.11. Remark. Theorem 1.10. b) and the principle of mathematical induction directly imply the following:

$$|z_1 z_2 ... z_n| = |z_1| \cdot |z_2| \cdot ... \cdot |z_n|. \blacksquare \qquad (1)$$

1.12. Theorem. If $zw = n$, then $z = n$ or $w = n$.

Proof. If $zw = n$, then Theorem 1.10 implies

$$|z| \cdot |w| = |zw| = |n| = 0.$$

But, $|z|$ and $|w|$ are real numbers, and thus $|z| = 0$ or $|w| = 0$, i.e. $z = n$ or $w = n$. ■

1.13. Theorem. If $z \neq n$ and $zw = zw_1$, then $w = w_1$.

Proof. The given condition $zw = zw_1$ implies that $-zw = -zw_1$. So,

$$n = zw - zw_1 = z(w - w_1).$$

Acording to Theorem 1.12, $z \neq n$ implies that

$$w - w_1 = n, \text{ i.e. } w = w_1. \blacksquare$$

1.14. Theorem. For each complex number $z \neq n$ there exists one and only one complex number w, denoted by $\frac{e}{z}$, so that $zw = e$ holds.

Proof. Firstly, we will prove the existence of the complex number $w = \frac{e}{z}$. Let $z = (a,b) \neq n$ be an arbitrary complex number. Let

$$w = (\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}),$$

So,

$$zw = (a,b) \cdot (\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}) = (1,0) = e.$$

The uniqueness is implied immideately by Theorem 1.13. ■

1.15. Theorem. If $z \neq n$, then for each complex number w there exists one and only one complex number u, so that $zu = w$ holds.

Proof. By Theorem 1.14, for any complex number $z \neq n$ there exists one and only one complex number $\frac{e}{z}$, so that $z \cdot \frac{e}{z} = e$ holds. Let $u = \frac{e}{z} \cdot w$ Thus we get a unique complex number u such that satisfies the following

$$zu = z \cdot \frac{e}{z} w = w. \blacksquare$$