Problems and solutions

Problem 1. Let ABC be a triangle and Q a point on the internal angle bisector of $\angle BAC$. Circle ω_1 is circumscribed to triangle BAQ and intersects the segment AC in point $P \neq C$. Circle ω_2 is circumscribed to the triangle CQP. Radius of the cirlce ω_1 is larger than the radius of ω_2. Circle centered at Q with radius QA intersects the circle ω_1 in points A and A_1. Circle centered at Q with radius QC intersects ω_1 in points C_1 and C_2. Prove $\angle A_1BC_1 = \angle C_2PA$.

(Matija Bucić)

Solution. From the conditions in the problem we have $|QC_1| = |QC_2|$ and $|QA| = |QA_1|$. Also as Q lies on the internal angle bisector of$\angle CAB$ we have $\angle PAQ = \angle QAB \implies |QP| = |QB|$.

Now noting from this that pairs of points A and A_1, C_1 and C_2, B and P are symmetric in line QS_1, where S_1 is the center of ω_1. We can directly conclude $\angle A_1BC_1 = \angle APC_2$ as these is the image of the angle in symmetry.

This way we have avoided checking many cases but there are many ways to prove this problem.

Problem 2. Let S be the set of positive integers. For any a and b in the set we have $GCD(a,b) > 1$. For any a, b and c in the set we have $GCD(a,b,c) = 1$. Is it possible that S has 2012 elements?

$GCD(x,y)$ and $GCD(x,y,z)$ stand for the greatest common divisor of the numbers x and y and numbers x, y and z respectively.

(Ognjen Stipetić)

Solution. There is such a set.

We will construct it in the following way: Let $a_1, a_2, \ldots a_{2012}$ equal to 1 in the begining. Then we take $\frac{2012\cdot 2011}{2}$ different prime numbers, and assign a different prime to every pair a_i, a_j (where $i \neq j$) and multiply them with this assigned number. (I.e. for the set of 4 elements we can take 2, 3, 5, 7, 11, 13, so S would be $\{2\cdot 3\cdot 5, 2\cdot 7\cdot 11, 3\cdot 7\cdot 13, 5\cdot 11\cdot 13\}$.

The construction works as we have multiplied any pair of numbers with some prime so the condition $gcd(a,b) > 1$ is satisfied for all a, b. As well as each prime divides exactly 2 primes so no three numbers a, b, c can have $gcd(a,b,c) > 1$.

Problem 3. Do there exist positive real numbers x, y and z such that

$$x^4 + y^4 + z^4 = 13,$$
$$x^3y^3z + y^3z^3x + z^3x^3y = 6\sqrt{3},$$
$$x^3yz + y^3zx + z^3xy = 5\sqrt{3}?$$

(Matko Ljulj)

Solution. Let's assume that such x, y, z exist. Let $a = x^2$, $b = y^2$, $c = z^2$. As well, let $A = a + b + c$, $B = ab + bc + ca$, $C = abc$. The upper system can be rewritten as:

$$a^2 + b^2 + c^2 = 13 \implies (a+b+c)^2 - 2(ab+bc+ca) = 13 \implies A^2 - 2B = 13$$
$$xyz(x^2y^2 + y^2z^2 + z^2x^2) = 6\sqrt{3} \implies \sqrt{C}B = 6\sqrt{3}$$
$$xyz(x^2 + y^2 + z^2) = 5\sqrt{3} \implies \sqrt{C}A = 5\sqrt{3}.$$

We can note that a, b and c are positive reals (They are not negaitve from the definition; and as $\sqrt{C}B = 6\sqrt{3}$ they are not 0).

When we cancel out $\sqrt{C}$ from the second and third equation we get $5B = 6A$. When we express B in terms of A and put int the first equation we get a quadratic equation

$$A^2 - \frac{12}{5}A - 13 = 0.$$

with solutions 5 and $-\frac{13}{5}$. As a, b and c are positive reals, and the sum must be positive so their sum is poistive real number as well. So $A = 5 \implies B = 6 \implies C = 3$.

By *AM-GM* inequality we get

$$\begin{aligned}\frac{ab+bc+ca}{3} &\geqslant \sqrt[3]{ab \cdot bc \cdot ca} \\ \iff \frac{B}{3} &\geqslant \sqrt[3]{C^2} \\ \iff \frac{6}{3} &\geqslant \sqrt[3]{9} \;/^3 \\ \iff 8 &\geqslant 9.\end{aligned}$$

so we reached a contradiction, thus such x, y, z don't exist.

Problem 4. Let k be a positive integer. At the European Chess Cup every pair of players played a game in which somebody won (there were no draws). For any k players there was a player against whom they all lost, and the number of players was the least possible for such k. Is it possible that at the Closing Ceremony all the participants were seated at the round table in such a way that every participant was seated next to both a person he won against and a person he lost against.

(Matija Bucić)

Solution. The answer is yes.

In this problem we could use graph theory terminology but as this problem was intended for younger students we shall avoid mentioning any specific graph theory terms.

Let's take the largest number of participants whom we can seat around the table as desired. If we have seated all the participants we are done. Otherwise there is a person not seated at the table. As well there is at least one person seated at the table so let's name it a.

WLOG we can assume that for each person seated at the table to his right there is a person he won against and to his left a person he lost against.

Denote by W the set of people who won against person a, and are not seated at the table. Similarly, let L denote the set of all people who lost against a and are not seated at the table.

Let's consider any person p from W. If person p lost against the left neighbour of a, then we could seat p in between a and his (former) left neighbour, which is a contradiction with the assumption that we have seated the maximal possible number of people. So p won against the left neighbour of a. Using similar deduction we conclude that p won against the next left neighbour as well etc. So p must have won against everybody seated at the table.

In the same way if we consider any person q from L and consider the right neighbour of a, we can conclude that q lost against every person seated at the table.

If some person r from W lost against some person s in L, then instead of seating a we can seat s and r respectively by which we would reach a contradiction to the number of people seated being maximal.

So we conclude that all the people in W won against all people not in W and all the people in L lost against all people not in L.

As there is a someone who is not seated either W or L is non-empty. If W is non-empty, we can consider the set W as an independent chess cup. It is a cup with smaller number of participants but still satisfying problem conditions which would be the contradiction with the fact that our starting cup is the smallest such cup.

As well if L is non-empty, the smaller cup made by people seated at the table and people in W also satisfies the problem conditions and gives us a contradiction.

So the only possibility is that both W and L are empty so indeed it is possible to seat everyone at such table.

2^{ND} European Mathematical Cup

7^{th} December 2013–15^{st} December 2013

Junior Category

Problems and Solutions

Problem 1. For a positive integer m let $m?$ be the product of first m prime numbers. Determine if there exist positive integers m and n with the following property:

$$m? = n(n+1)(n+2)(n+3).$$

(Matko Ljulj)

Solution. Such numbers don't exist.

Let's assume the contrary i.e. there are such m and n.

We can note that there is only one prime divisible by 2 and that it 2 itself thus $m?$ isn't divisible by 4. On the other hand, the product $n(n+1)(n+2)(n+3)$ is product of 4 consecutive integers so two of them are even making the product divisible by 4.

Thus equality $m? = n(n+1)(n+2)(n+3)$ gives us a contradiction as LHS is not divisible by 4 while RHS is.

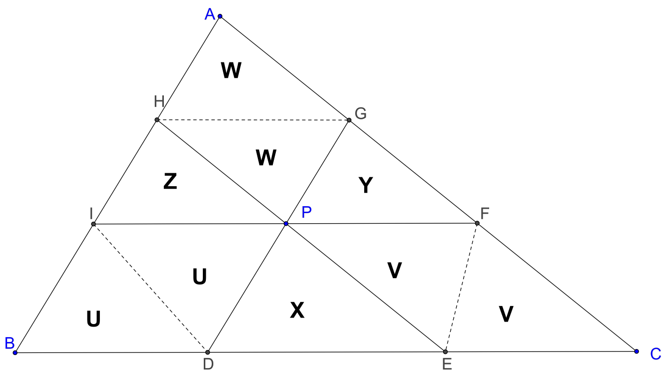
Problem 2. Let P be a point inside a triangle ABC. A line through P parallel to AB meets BC and CA at points L and F, respectively. A line through P parallel to BC meets CA and BA at points M and D respectively, and a line through P parallel to CA meets AB and BC at points N and E respectively. Prove

$$(PDBL) \cdot (PECM) \cdot (PFAN) = 8 \cdot (PFM) \cdot (PEL) \cdot (PDN),$$

where (XYZ) and $(XYZW)$ denote the area of the triangle XYZ and the area of quadrilateral $XYZW$.

(Steve Dinh)

Solution.



Let's denote the areas as on the sketch.

The problem is equivalent to

$$U \cdot V \cdot W = X \cdot Y \cdot Z.$$

Let x and y be lengths of altitudes from I and D in the triangle BID and let a and b be lenghts of sides BI and BD. We can deduce

$$X = (PED) = \frac{1}{2} \cdot a \cdot y \cdot \frac{BC}{BA},$$

$$Z = (PIH) = \frac{1}{2} \cdot b \cdot x \cdot \frac{BA}{BC} \text{ and}$$

$$U = (BID) = \frac{1}{2} \cdot a \cdot x = \frac{1}{2} \cdot b \cdot y$$

This gives $U^2 = X \cdot Z$. Analogously we get $W^2 = Y \cdot Z$ and $V^2 = X \cdot Y$. Multiplying all three equalities we get the desired equation.

Second solution. Let's denote the areas of triangles PEL, PFM, PDN as P_A, P_B, P_C respectively and let's denote the areas of quadrilaterals $PFAN$, $PDBL$, $PECM$ as Q_A, Q_B, Q_C respectively. We want to prove $Q_AQ_BQ_C = 8P_AP_BP_C$. Triangles PEL, PFM, and PDN are similar to the triangle ABC (they have respective pairs of sides on parallel lines). Let's denote the respective similarity coefficients as k_A, k_B, k_C. As triangles PEL, PFM, and PDN are in the interior of ABC, all those coefficients are less than 1.
Triangle ENB is similar to the triangle ABC. Its similarity coefficient is

$$\frac{EN}{AC} = \frac{EF + FN}{AC} = \frac{EF}{AC} + \frac{FN}{AC} = k_A + k_C.$$

From all these similarity relations we get area relations. Namely:

$$P_A : P_B = (P_A : (ABC)) : (P_B : (ABC)) = \left(\frac{k_A}{k_B}\right)^2 \implies P_A = \left(\frac{k_A}{k_B}\right)^2 P_B,$$
$$P_C : P_B = (P_C : (ABC)) : (P_B : (ABC)) = \left(\frac{k_C}{k_B}\right)^2 \implies P_C = \left(\frac{k_C}{k_B}\right)^2 P_B.$$

Using this we get:

$$(P_A + P_C + Q_B) : P_B = (ENB) : (PFM) = (k_A + k_C)^2 : (k_B)^2$$
$$\implies P_A + P_C + Q_B = \frac{k_A^2 + 2k_Ak_C + k_C^2}{k_B^2} P_B = \frac{k_A^2}{k_B^2} P_B + \frac{2k_Ak_C}{k_B^2} P_B + \frac{k_C^2}{k_B^2} P_B = P_A + \frac{2k_Ak_C}{k_B^2} P_B + P_C$$
$$\implies Q_B = \frac{2k_Ak_C}{k_B^2} P_B.$$

Similary by the same process applied to FLC and MDA we get $Q_C = \frac{2k_Bk_A}{k_C^2} P_C$ i $Q_A = \frac{2k_Ck_B}{k_A^2} P_A$. Multiplying what we got we have

$$Q_AQ_BQ_C = \frac{2k_Ck_B}{k_A^2} P_A \frac{2k_Ak_C}{k_B^2} P_B \frac{2k_Bk_A}{k_C^2} P_C = 8\frac{k_A^2k_B^2k_C^2}{k_A^2k_B^2k_C^2} P_AP_BP_C = 8P_AP_BP_C,$$

Q.E.D.

Problem 3. We are given a combination lock consisting of 6 rotating discs. Each disc consists of digits $0, 1, 2, \ldots, 9$, in that order (after digit 9 comes 0). Lock is opened by exactly one combination. A move consists of turning one of the discs one digit in any direction and the lock opens instantly if the current combination is correct. Discs are initially put in the position 000000, and we know that this combination is not correct.

a) What is the least number of moves necessary to ensure that we have found the correct combination?
b) What is the least number of moves necessary to ensure that we have found the correct combination, if we know that none of the combinations 000000, 111111, 222222, ..., 999999 is correct?

(Ognjen Stipetić, Grgur Valentić)

Solution. We will solve the subproblems seperately.

a) In order to ensure that we have discovered the code we need to check all but one of the combinations (as otherwise all unchecked codes can be the correct combination). Total number of combinations is 10^6 (as each of the 6 discs consists of 10 digits). As we are given that 000000 is not the correct combination we require at least $10^6 - 2$ moves. We will now prove that there is a sequence of $10^6 - 2$ moves each checking a different combination. We will prove this by induction on the number of wheels where the case $n = 6$ is given in the problem.

   Claim: For a lock of n wheels and for any starting combination of the wheels $(a_1a_2 \ldots a_n)$ there is a sequence of moves checking all 10^6 combinations exactly once, for all $n \in \mathbb{N}$.

   Basis: For $n = 1$ and for the starting combination (a), we consider the sequence of moves

$$a \to a + 1 \to a + 2 \to \ldots \to 9 \to 0 \to 1 \to \ldots \to a - 1$$

Assumption: The induction claim is valid for some $n \in \mathbb{N}$.
Step: We will prove that the claim holds for $n+1$ as well. We consider an arbitrary starting state $(a_1 a_2 \dots a_n a_{n+1})$. By the induction hypothesis there is a sequence of moves such that starting from this state we can check all the states showing a_{n+1} on the last disc. Let this sequence of moves end with the combination $(b_1 b_2 \dots b_n a_{n+1})$.
Now we make the move $(b_1 b_2 \dots b_n a_{n+1}) \to (b_1 b_2 \dots b_n a_{n+1} + 1)$ (if a_{n+1} is 9, then we turn the disc to show 0).
We continue in the same way applying the induction hypothesis on first n discs and the rotation the $n+1$-st disc. This way we get the sequence of moves

$$\begin{aligned}(a_1 a_2 \dots a_n a_{n+1}) &\to (b_1 b_2 \dots b_n a_{n+1}) \to (b_1 b_2 \dots b_n a_{n+1} + 1) \\ &\to (c_1 c_2 \dots c_n a_{n+1} + 1) \to (c_1 c_2 \dots c_n a_{n+1} + 2) \\ &\dots \\ &\to (j_1 j_2 \dots j_n a_{n+1} - 2) \to (j_1 j_2 \dots j_n a_{n+1} - 1).\end{aligned}$$

This sequence checks each combination exactly once finishing the induction and proving our claim.

b) As in the a) part, we conclude that we have to check all the combinations apart from 000000, 111111, ..., 999999 and we can be sure as to what is the solution before the move checking the last combination.
We denote the combination as *black* if the sum of its digits is even and *white* if that sum is odd. We can notice that all the combinations 000000, 111111, ..., 999999 are black and by each move we swap the color of the current combination.
Number of black combinations all of which we need to check at least once is $\frac{10^6}{2} - 10$ while number of such white combinations is $\frac{10^6}{2}$.
As we are checking white combinations every second move, in order to check all $\frac{10^6}{2}$ white combination swe need at least $2\frac{10^6}{2} - 1 = 10^6 - 1$ moves, thus we need at least $10^6 - 2$ moves to find the correct combination.
An example doing this in $10^6 - 2$ moves has been given in part a).

Problem 4. Let a, b, c be positive real numbers satisfying

$$\frac{a}{1+b+c} + \frac{b}{1+c+a} + \frac{c}{1+a+b} \geqslant \frac{ab}{1+a+b} + \frac{bc}{1+b+c} + \frac{ca}{1+c+a}.$$

Prove

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + a + b + c + 2 \geqslant 2(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}).$$

(Dimitar Trenevski)

Solution. We start with the given condition:

$$\begin{gathered}\frac{a}{1+b+c} + \frac{b}{1+c+a} + \frac{c}{1+a+b} \geqslant \frac{ab}{1+a+b} + \frac{bc}{1+b+c} + \frac{ca}{1+c+a} \iff \\ \frac{a+ab+bc}{1+b+c} + \frac{b+bc+ba}{1+c+a} + \frac{c+ca+cb}{1+a+b} \geqslant \frac{ab+ac+bc}{1+a+b} + \frac{bc+ab+bc}{1+b+c} + \frac{ca+bc+ab}{1+c+a} \iff \\ \frac{a(1+b+c)}{1+b+c} + \frac{b(1+c+a)}{1+c+a} + \frac{c(1+a+b)}{1+a+b} \geqslant \frac{ab+bc+ca}{1+a+b} + \frac{ab+bc+ca}{1+b+c} + \frac{ab+bc+ca}{1+c+a} \iff \\ a+b+c \geqslant (ab+bc+ca)\left(\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a}\right).\end{gathered}$$

Now using *Cauchy-Schwarz* inequality we get:

$$\left(\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a}\right)(c(1+a+b) + a(1+b+c) + b(1+c+a)) \geqslant (\sqrt{a} + \sqrt{b} + \sqrt{c})^2.$$

Combining the last two inequalities we get:

$$\begin{gathered}(a+b+c)(a+b+c+2(ab+bc+ca)) \geqslant \\ \geqslant (ab+bc+ca)\left(\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a}\right)(a+b+c+2(ab+bc+ca)) = \\ = (ab+bc+ca)\left(\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a}\right)(c(1+a+b) + a(1+b+c) + b(1+c+a)) \geqslant \\ \geqslant (ab+bc+ca)(\sqrt{a} + \sqrt{b} + \sqrt{c})^2,\end{gathered}$$

which now by some algebraic manipulation gives:

$$(a+b+c)(a+b+c+2(ab+bc+ca)) \geqslant (ab+bc+ca)(\sqrt{a}+\sqrt{b}+\sqrt{c})^2 \iff$$
$$(a+b+c)^2+2(a+b+c)(ab+bc+ca) \geqslant (ab+bc+ca)(a+b+c+2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})) \iff$$
$$(a^2+b^2+c^2)+(2(a+b+c)+2)(ab+bc+ca) \geqslant (ab+bc+ca)(a+b+c+2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})) \iff$$
$$\frac{a^2+b^2+c^2}{ab+bc+ca}+a+b+c+2 \geqslant 2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}),$$

where the last inequality is exactly the one we wanted to prove.

Time allowed: 240 minutes.

Each problem is worth 10 points.

Calculators are not allowed.

1$^{\text{ST}}$ EUROPEAN MATHEMATICAL CUP

24^{th} November 2012–1^{st} December 2012

Senior Category

Problems and solutions

Problem 1. Find all positive integers a, b, n and prime numbers p that satisfy

$$a^{2013} + b^{2013} = p^n.$$

(Matija Bucić)

First solution. Let's denote $d = D(a, b), x = \frac{a}{d}, y = \frac{b}{d}$. With this we get

$$d^{2013}(a^{2013} + b^{2013}) = p^n.$$

So d must be a power of p, so let $d = p^k, k \in \mathbb{N}_0$. We can divide the equality by p^{2013k}. Now let's denote $m = n - 2013k, A = x^{671}, B = y^{671}$. So we get

$$A^3 + B^3 = p^m,$$

and after factorisation

$$(A + B)(A^2 - AB + B^2) = p^m.$$

(From the definition, A and B are coprime.)

Let's observe the case when some factor is 1: $A + B = 1$ is impossible as both A and B are positive integers. And $A^2 - AB + B^2 = 1 \Leftrightarrow (A - B)^2 + AB = 1 \Leftrightarrow A = B = 1$, so we get a solution $a = b = 2^k, n = 2013k + 1, p = 2, \forall k \in \mathbb{N}_0$.

If both factors are larger than 1 we have

$$\begin{aligned} p &\mid A + B \\ p &\mid A^2 - AB + B^2 = (A + B)^2 - 3AB \\ \implies p &\mid 3AB. \end{aligned}$$

If $p \mid AB$, in accordance with $p \mid A + B$ we get $p \mid A$ and $p \mid B$, which is in contradiction with A and B being coprime. So, $p \mid 3 \implies p = 3$.

Now we are left with 2 cases:

- First case: $A^2 - AB + B^2 = 3 \Leftrightarrow (A - B)^2 + AB = 3$ – so the only possible solutions are $A = 2, B = 1$ i $A = 1, B = 2$, but this turns out not to be a solution as $2 = x^{671}$ does not have a solution in positive integers.
- Second case: $3^2 \mid A^2 - AB + B^2$ – then we have:

$$\begin{aligned} 3 \mid A + B &\implies 3^2 \mid (A + B)^2 \\ 3^2 \mid A^2 - AB + B^2 &= (A + B)^2 - 3AB \\ &\implies 3^2 \mid 3AB \\ &\implies 3 \mid AB. \end{aligned}$$

  And as we have already commented the case $p \nmid AB \implies$ doesn't have any solutions.

So all the solutions are given by

$$a = b = 2^k, n = 2013k + 1, p = 2, \forall k \in \mathbb{N}_0.$$

Second solution. As in the first solution, we take the highest common factor of a and b (which must be of the form p^k). Factorising the given equality we get

$$(x+y)(x^{2012}-x^{2011}y+x^{2010}y^2-\cdots-xy^{2011}+y^{2012})=p^m.$$

(We're using the same notation as in the first solution.) Denote the right hand side factor by A. As x and y are natural numbers, we have $x+y>1 \implies p \mid x+y$. So $p\nmid x$ and $p \nmid y$ (as x and y are coprime). Now by applying LTE (Lifting the Exponent Lemma):

$$\nu_p(x^{2013}+y^{2013})=\nu_p(x+y)+\nu_p(2013)$$

Now we know $\nu_p(2013)=0$ fo all primes p except $3,11,61$, and in the remaining cases $\nu_p(2013)=1$. Note $A=1$ and $(x,y)=(1,1)$ and $A>61$ for $(x,y)\neq(1,1)$. This inequality holds because for $(x,y)\neq(1,1)$ (WLOG $x\geqslant y$), we can write A as

$$x^{2011}(x-y)+x^{2009}y^2(x-y)+\cdots+xy^{2010}(x-y)+y^{2012},$$

which is greater than 61 in cases $x>y$ and $y\neq 1$.

- If $\nu_p(2013)=1 \implies \nu_p(A)=1 \implies A\in\{3,11,61\}$ which is clearly impossible.
- If $\nu_p(2013)=0 \implies \nu_p(A)=0 \implies A=1 \implies (x,y)=(1,1)$, so we get a solution

$$a=b=2^k, n=2013k+1, p=2, \forall k\in\mathbb{N}_0.$$

Problem 2. Let ABC be an acute triangle with orthocenter H. Segments AH and CH intersect segments BC and AB in points A_1 and C_1 respectively. The segments BH and A_1C_1 meet at point D. Let P be the midpoint of the segment BH. Let D' be the reflection of the point D in AC. Prove that quadrilateral $APCD'$ is cyclic.

(Matko Ljulj)

First solution. We shall prove that D is the orthocenter of triangle APC. From that the problem statement follows as

$$\begin{aligned}\angle AD'C=\angle ADC=180^\circ-\angle DAC-\angle DCA=(90^\circ-\angle DAC)+(90^\circ-\angle DCA)=\\ =\angle PCA+\angle PAC=180^\circ-\angle APC.\end{aligned}$$

We can note that quadrilateral BA_1HC_1 is cyclic. Lines BA_1 and C_1H intersect in C, lines BC_1 and A_1H intersect in A, lines BH and C_1A_1 intersect in D, and point P is the circumcenter of BA_1HC_1. So by the corollary of the Brocard's theorem point D is indeed the orthocenter of triangle APC as desired.

Second solution. Denote by B_1 the orthogonal projection of B on AC. By cyclic quadrilaterals $B_1C_1PA_1$ (Euler's circle), HA_1CB_1, AC_1A_1C and C_1HB_1A we get the following equations:

$$\begin{aligned}\angle A_1PB_1=\angle DC_1B_1\\ \angle A_1B_1P=\angle A_1CC_1=\angle A_1AC_1=\angle DB_1C_1.\end{aligned}$$

From these equalities we get that triangles B_1PA_1 and B_1C_1D are similar, which implies

$$\frac{|B_1D|}{|B_1A_1|}=\frac{|B_1C_1|}{|B_1P|}\implies |B_1A_1|\cdot|B_1C_1|=|B_1D|\cdot|B_1P|.$$

Analogously, using cyclic quadrilateral ABA_1B_1 and C_1BCB_1 we get the following angle equations:

$$\begin{aligned}\angle B_1AC_1=180^\circ-\angle B_1A_1B=\angle B_1A_1C\\ \angle AB_1C_1=180^\circ-\angle C_1B_1C=\angle CBA=180^\circ-\angle A_1B_1A=\angle A_1B_1C.\end{aligned}$$

From these equalities we get that triangles B_1AC_1 and B_1A_C are similar so

$$\frac{|B_1C_1|}{|B_1C|}=\frac{|AB_1|}{|A_1B_1|}\implies |B_1A_1|\cdot|B_1C_1|=|B_1A|\cdot|B_1C|.$$

Thus we get $|B_1D'|\cdot|B_1P|=|B_1D|\cdot|B_1P|=|B_1A_1|\cdot|B_1C_1|=|B_1A|\cdot|B_1C|$ so by the reverse of the power of the point theorem the quadrilateral $APCD'$ is cyclic as desired.

Problem 3. Prove that the following inequality holds for all positive real numbers a, b, c, d, e and f:

$$\sqrt[3]{\frac{abc}{a+b+d}}+\sqrt[3]{\frac{def}{c+e+f}}<\sqrt[3]{(a+b+d)(c+e+f)}.$$

(Dimitar Trenevski)

Solution. The inequality is equivalent to

$$\sqrt[3]{\frac{abc}{(a+b+d)^2(c+e+f)}}+\sqrt[3]{\frac{def}{(a+b+d)(c+e+f)^2}}<1.$$

By AM-GM inequality we have

$$\sqrt[3]{\frac{abc}{(a+b+d)^2(c+e+f)}}\leqslant\frac{1}{3}\left(\frac{a}{a+b+d}+\frac{b}{a+b+d}+\frac{c}{c+e+f}\right),$$

$$\sqrt[3]{\frac{def}{(a+b+d)(c+e+f)^2}}\leqslant\frac{1}{3}\left(\frac{d}{a+b+d}+\frac{e}{c+e+f}+\frac{f}{c+e+f}\right).$$

Adding the inequalities we get

$$\sqrt[3]{\frac{abc}{(a+b+d)^2(c+e+f)}}+\sqrt[3]{\frac{def}{(a+b+d)(c+e+f)^2}}\leqslant\frac{1}{3}\left(\frac{a+b+d}{a+b+d}+\frac{c+e+f}{c+e+f}\right)=\frac{2}{3}<1,$$

as desired.

Problem 4. Olja writes down n positive integers $a_1, a_2, \ldots, a_n$ smaller than p_n where p_n denotes the n-th prime number. Oleg can choose two (not necessarily different) numbers x and y and replace one of them with their product xy. If there are two equal numbers Oleg wins. Can Oleg guarantee a win?

(Matko Ljulj)

Solution. For $n = 1$, Oleg won't be able to write 2 equal numbers on the board as there will be only one number written on the board. We shall now consider the case $n > 2$.

Let's note that as all the numbers are strictly smaller than p_n we have all their prime factors are from the set $\{p_1, p_2, \ldots, p_{n-1}\}$, so there are at most $n-1$ of them in total. We will represent each number $a_1, a_2, \ldots, a_n$ by the ordered $(n-1)$-tuple of non-negative integers in the following way if $a_i = p_1^{\alpha_{i,1}} \cdot p_2^{\alpha_{i,2}} \cdot \ldots \cdot p_{n-1}^{\alpha_{i,(n-1)}}$, then we assign $v_i = (\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,(n-1)})$, for all $i \in \{1, 2, \ldots, n\}$.

Let's consider the following system of equations:

$$\begin{gathered}\alpha_{1,1}x_1+\alpha_{2,1}x_2+\cdots+\alpha_{n,1}x_n=0\\ \alpha_{1,2}x_1+\alpha_{2,2}x_2+\cdots+\alpha_{n,2}x_n=0\\ \cdots\\ \alpha_{1,(n-1)}x_1+\alpha_{2,(n-1)}x_2+\cdots+\alpha_{n,(n-1)}x_n=0\end{gathered}$$

There is a trivial solution $x_1 = x_2 = \cdots = x_n = 0$. But as this system has less equalities than variables we can deduce that it has infinitely many solutions in the set of rational numbers (as all the coefficients are rational). Let $(y_1, y_2, \ldots, y_n)$ be a not trivial solution (so the solution in which not all of y_i equal 0). Then we can rewrite the initial system using $a_1, a_2, \ldots, a_n$:

$$\prod_{i=1}^{n}a_i^{y_i}=\prod_{i=1}^{n}p_1^{\alpha_{i,1}y_i}\cdot p_2^{\alpha_{i,2}y_i}\cdot\ldots\cdot p_{n-1}^{\alpha_{i,(n-1)}y_i}=\prod_{j=1}^{n-1}p_j^{\alpha_{1,j}y_1+\alpha_{2,j}y_2+\cdots+\alpha_{n,j}y_n}=\prod_{j=1}^{n-1}p_j^0=1$$

$$\Longrightarrow\prod_{i=1}^{n}a_i^{y_i}=1.$$

Considering the numbers $y_1, y_2, \ldots, y_n$ as rational numbers in which the respective nominator and denominator are coprime, Denote by L the lowest common multiplier of their denominators. Taking the L-th power of the upper equality we get integer exponents in the upper equation (which don't have a common factor). Furthermore, WLOG we can assume that $a_1, a_2, \ldots, a_k$ are those elements a_i whose exponents are negative and numbers $a_{k+1}, a_{k+2}, \ldots, a_{k+l}$ are those elements with postivie exponent (for some $k,l \in \mathbb{N}$, $k+l \leqslant n$). Then, when we shift all a_i-s with negative exponent to the opposite side of the equation and when those with zero exponent get ruled out we get that the following equality

$$\prod_{i=1}^{k} a_i^{r_i} = \prod_{i=k+1}^{l} a_i^{r_i} \tag{1}$$

holds for some positive integers $r_1, r_2, \ldots, r_{k+l}$ for which $D(r_1, r_2, \ldots, r_{k+l}) = 1$ and for some numbers $a_1, a_2, \ldots, a_{k+l}$. (We can note that there is at least one number a_i on both sides of the equality otherwise we have only ones on the board.)

We shall prove that there is a sequence of transformations by which using this relation we will get two equal numbers among $a_1, a_2, \ldots a_n$.

Lemma 1. *Let $(a,b) \in \mathbb{N}^2$ and $(x_1,x_2) \in \mathbb{N}^2$ be such that $GCD(x_1,x_2) = 1$. Then there exists a sequence of transformations which replaces the numbers (a,b) with (a',b'), where one of these numbers a', b' is equal to $a^{x_1}b^{x_2}$.*

Proof. We'll prove this by induction on $x_1 + x_2$, for all $(a,b) \in \mathbb{N}^2$. As the basis consider $x_1 + x_2 = 2 \implies x_1 = x_2 = 1$. The number ab we can get by applying transformation $(a,b) \to (a,ab)$.

Let's assume that the claim holds for all (x_1,x_2) such that $x_1 + x_2 < n$, and for all (a,b). Let's take some numbers (x_1,x_2) such that $x_1 + x_2 = n$ and some arbitrary numbers (a,b). If $x_1 = x_2$ is satisfied, since x_1 and x_2 are coprime, we could conclude that both numbers are equal to 1, but we have already proved this case in basis. Let's assume $x_1 \neq x_2$. WLOG $x_1 > x_2$. Then we apply the transformation $(a,b) \to (a,ab)$, and then apply the induction hypothesis on numbers (a,ab) and $(x_1 - x_2, x_2)$:

$$(a,b) \to (a,ab) \to (\gamma, a^{x_1-x_2}(ab)^{x_2}) = (\gamma, a^{x_1}b^{x_2}),$$

where γ is some positive integer, what we wanted to prove. □

Lemma 2. *Let $k \in \mathbb{N}$, $(b_1, b_2, \ldots b_k) \in \mathbb{N}^k$ and $(x_1, x_2, \ldots x_k) \in \mathbb{N}^k$. Then there exists sequence of transformations which instead of numbers $(b_1, b_2, \ldots b_k)$ writes down numbers $(b'_1, b'_2, \ldots b'_k)$ such that one of those numbers is equal to*

$$(b_1^{x_1} b_2^{x_2} \cdots b_k^{x_k})^{\frac{1}{d}},$$

where d denotes greatest common divisor of numbers $x_1, x_2, \ldots x_k$.

Proof. Intuitively, this lemma is just *Lemma 1* repeated $(k-1)$ times.

We'll prove this by induction on k, for all $b_1, b_2, \ldots b_k$ and $x_1, x_2, \ldots x_k$. In the basis, for $k = 1$, it holds $d = x_1$, so it we don't have to do any transformation to reach desired situation.

Let's assume that the claim holds for some $k \in \mathbb{N}$. Let's take arbitrary $(b_1, b_2, \ldots b_k, b_{k+1})$ and $(x_1, x_2, \ldots x_k, x_{k+1})$. Then we apply *Lemma 1* on numbers (b_k, b_{k+1}) and (x'_k, x'_{k+1}), where $x'_k = \frac{x_k}{d_1}$, $x'_{k+1} = \frac{x_{k+1}}{d_1}$, $d_1 = GCD(x_k, x_{k+1})$, and then we apply the induction hypothesis on numbers $(b_1, b_2, \ldots b_k^{x'_k} b_{k+1}^{x'_{k+1}})$ and $(x_1, x_2, \ldots x_{k-1}, d_1)$:

$$(b_1, b_2, \ldots b_k, b_{k+1}) \to (b_1, b_2, \ldots b_{k-1}, \gamma_k, b_k^{x'_k} b_{k+1}^{x'_{k+1}}) \to (\gamma_1, \gamma_2, \ldots, \gamma_k, (b_1^{x_1} b_2^{x_2} \cdots b_{k-1}^{x_{k-1}} (b_k^{x'_k} b_{k+1}^{x'_{k+1}})^{d_1})^{\frac{1}{d_2}}),$$

where $\gamma_1, \gamma_2, \ldots, \gamma_k$ are some positive integers and $d_2 = GCD(x_1, x_2, \ldots x_{k-1}, d) = GCD(x_1, x_2, \ldots x_{k-1}, x_k, x_{k+1}) = d$. Notice that last number in upper relation is the one we wanted to get. □

Lemma 3. *Let $(a,b) \in \mathbb{N}^2$ and $(x_1,x_2) \in \mathbb{N}^2$ such that $GCD(x_1,x_2) = 1$. Then there exists sequence of transformations which instead of numbers (a,b) writes down numbers (a',b') for which it is satisfied $a'/b' = a^{x_1}/b^{x_2}$.*

Proof. We'll prove this by induction on $x_1 + x_2$, for all $(a,b) \in \mathbb{N}^2$. In the basis is $x_1 + x_2 = 2 \implies x_1 = x_2 = 1$, so we don't have to do any transformation to reach desired situation.

Ler's assume that the claim hold for all (x_1,x_2) such that $x_1 + x_2 < n$, and for all (a,b). Let's take some numbers (x_1,x_2) such that $x_1 + x_2 = n$ and arbitrary numbers (a,b).

- If one of the numbers x_1 and x_2 is even (WLOG x_1 is even): we apply tranformation $(a,b) \to (a^2,b)$, and then we apply induction hypothesis on numbers (a^2,b) and $(\frac{x_1}{2}, x_2)$.
- Both numbers x_1 and x_2 are odd, and they are equal: then they are both equal to 1, which we have already solved in the basis.
- Numbers x_1 and x_2 are odd and distinct (WLOG $x_1 > x_2$): we make following transformations $(a,b) \to (a,ab) \to (a^2,ab)$, and then we apply induction hypothesis on numbers (a^2,ab) and $(\frac{x_1+x_2}{2}, x_2)$:

$$(a,b) \to (a,ab) \to (a^2,ab) \to (c \cdot (a^2)^{\frac{x_1+x_2}{2}}, c \cdot (ab)^{x_2}) = ((a^{x_2}c) \cdot a^{x_1}, (a^{x_2}c) \cdot b^{x_2}),$$

where c is some positive integer, what we wanted to prove. □

In the equality (1), let $d_1 = GCD(r_1, r_2, \ldots, r_k)$, $d_2 = GCD(r_{k+1}, r_{k+2}, \ldots, r_{k+l})$, $z_i = \frac{r_i}{d_1}$, $\forall i \in \{1, 2, \ldots, k\}$, $z_i = \frac{r_i}{d_2}$, $\forall i \in \{k+1, k+2, \ldots, k+l\}$. As well let A be the left hand side of the equality (1), and let B be the right hand side. Let $A' = A^{\frac{1}{d_1}}$ and $B' = B^{\frac{1}{d_2}}$. We want to do such transformations that we get x i y which will have same ratio as A and B.

If we apply *Lemma 2* on the numbers $(a_1, a_2, \ldots, a_k)$ and $(z_1, z_2, \ldots, z_k)$; we get (among other numbers we get) the number A'. As well applying the same lemma on the numbers $(a_{k+1}, a_{k+2}, \ldots, a_{k+l})$ and $(z_{k+1}, z_{k+2}, \ldots, z_{k+l})$, we will get the number B' on the board.

Numbers d_1 and d_2 are coprime (otherwise there would be some prime p which would divide d_1 and d_2 which would imply it divides $r_1, r_2, \ldots, r_{k+l}$ as well which is in contradiction to the assumption they do not have a common factor). So we can apply *Lemma 3* on the numbers (A', B') and (d_1, d_2). Now we get two numbers with the same ratio as A i B. But as by (1) we have $A = B$, we get 2 equal numbers on the board.

Thus Oleg can guarantee a win for any $n > 1$.

Comment: We can get to the relation (1) by concluding that the set $\{v_1, v_2, \ldots, v_n\}$ is linearly dependant subset of $(n-1)$-dimensional space $\mathbb{Q}^{n-1}$.

2$^{\text{ND}}$ European Mathematical Cup

7th December 2013–15st December 2013

Senior Category

Problems and Solutions

Problem 1. In each field of a table there is a real number. We call such $n \times n$ table *silly* if each entry equals the product of all the number in the neighbouring fields.

a) Find all 2×2 silly tables.

b) Find all 3×3 silly tables.

(Two fields of a table are neighbouring if they share a common side.) *(Borna Vukorepa)*

Solution. We solve the subproblems separately.

a) Denote the numbers in the table as on the picture:

a	b
c	d

By the problem condition we have the following:

$$a = bc$$
$$b = ad$$
$$c = ad$$
$$d = bc.$$

From here we can see $a = bc = d$ and $b = ad = c$. When we apply this to the upper relations we get $a = b^2$ and $b = a^2$ and so $a = b^2 = a^4 \iff a(a-1)(a^2+a+1) = 0$. The real solutions to this problem are $a = 0$ and $a = 1$. Now we can see that all 2×2 silly tables are those with all element equal and furthermore equal to zero or one.

b) Denote by a, b, c, d the elements in the table which have exactly three neighbours. We denote the remaining elements in terms of these and get the following table:

ab	a	ad
b	$abcd$	d
bc	c	cd

Let's assume that $abcd = 0$. This implies that the middle element is zero which further implies all its neighbours are zero and consequently every element in the table is zero. And thus only silly table under in this case is all zeros table.

Now assume that $abcd \neq 0$, i.e. none of the table elements is equal to zero. Using the remaining conditions we get:

$$a = (ab)(abcd)(ad) = a^3b^2d^2c \iff a^2b^2d^2c = 1,$$

Analogously we get $a^2b^2c^2d = 1$, $a^2c^2d^2b = 1$ i $b^2c^2d^2a = 1$ (we are allowed to divide by a, b, c, d as they are all non-zero). Equating the LHSs of these equations we get $a = b = c = d$. Inserting this in any of these equations we get $a^7 = 1 \implies a = 1$.

Thus all 3×3 silly tables are all ones and all zeros tables.

Problem 2. *Palindrome* is a sequence of digits which doesn't change if we reverse the order of its digits. Prove that a sequence $(x_n)_{n=0}^{\infty}$ defined as

$$x_n = 2013 + 317n$$

contains infinitely many numbers with their decimal expansions being palindromes.

(Stijn Cambie)

First solution. We will prove the following lemma providing two proofs:

Lema 1. *There is infinitely many numbers divisible with* 317 *with their decimal expansions consisting only of ones.*

Proof. Considering the sequence 1, 11, 111, ... consisting of infinitely many numbers. This numbers have some residues modulo 317. By The Pigeonhole Principle there are at least two numbers in this sequence with the same residue modulo 317. Let the smaller of these two have l digits and larger k. Their difference is

$$\underbrace{111\ldots1}_{k \text{ times}} - \underbrace{111\ldots1}_{l \text{ times}} = \underbrace{111\ldots1}_{(k-l) \text{ times}}\ \underbrace{000\ldots0}_{l \text{ times}}$$

divisible by 317. It will also remain divisible by 317 if we divide it by 10^l (as 10 and 317 are coprime). This way we get a number consisting only of ones divisible by 317. Let's denote the number of its digits by k. We get infinitely many such numbers by considering numbers consisting of k, $2k$, $3k$, ... ones. □

Proof. As 317 is prime, and as it is coprime with 10 by Fermat's Little Theorem

$$10^{316} \equiv 1\ (mod\ 317) \implies 317 \mid 10^{316m} - 1, \forall m \in \mathbb{Z}, m \geqslant 1.$$

As 9 is coprime with 317 as well, numbers of the form $\frac{1}{9}(10^{316m} - 1)$, $m \in \mathbb{Z}, m \geqslant 1$ have the property we desire. □

Continuing with the solution we can note that some integer m is in the sequence $(x_n)_{n=0}^{\infty}$ if and only if $m \geqslant 2013$ and $m \equiv 2013 \equiv 111\ (mod\ 317)$. Let $(y_n)_{n=0}^{\infty}$ be a sequence of infinitely many positive integers with their decimal expansions consisting only of ones and each being divisible by 317 (we are using our lemma here). Now numbers

$$1000y_n + 111$$

are in the sequence (as they have the remainder 111 modulo 317) and their decimal expansions are palindromes. Thus there is infinitely many members of the sequence $(x_n)_{n=0}^{\infty}$ whose decimal expansions are palindromes.

Second solution. We will prove the generalised version of the problem for the sequence $(x_n)_{n=0}^{\infty}$ defined as $x_n = a+nb$, where a, b are arbitrary positive integers with the property that b is coprime with 10. The problem is a special case of this for $a = 2013$ i $b = 317$.

We define the sequence $(y_n)_{n=0}^{\infty}$ in the following way: $y_n = 10^{n\varphi(b)}$. Using The Euler's Theorem, $y_n \equiv 1\ (mod\ b)$. Considering the number $1 + y_n + y_n^2 + \ldots y_n^{a-1}$, its decimal expansion is:

$$1\ \underbrace{000\ldots0}_{n\varphi(b)-1 \text{ times}}\ 1\ \underbrace{000\ldots0}_{n\varphi(b)-1 \text{ times}}\ \ldots\ 1\ \underbrace{000\ldots0}_{n\varphi(b)-1 \text{ times}}\ 1$$

where the digit one is repeated a times. It is clear now that the decimal expansion of this number is a palindrome. On the other hand $1 + y_n + y_n^2 + \ldots y_n^{a-1} \equiv 1 + 1 + \ldots 1 = a\ (mod\ b)$, so this number is in the sequence $(x_n)_{n=0}^{\infty}$, for each number n. Thus we have found infinitely many members of the sequence $(x_n)_{n=0}^{\infty}$ with their decimal expansions being palindromes as we wanted.

Problem 3. We call a sequence of n digits one or zero a *code*. Subsequence of a code is a *palindrome* if it is the same after we reverse the order of its digits. A palindrome is called *nice* if its digits occur consecutively in the code.*(Code* (1101) *contains 10 palindromes, of which 6 are nice.)*

a) What is the least number of palindromes in a code?
b) What is the least number of nice palindromes in a code?

(Ognjen Stipetić)

Solution. We will consider the two subproblems separately:

a) Consider any code. Assume there is k digits one and $n-k$ digits zero. We now transform this code into

$$\underbrace{111\ldots1}_{k \text{ puta}}\underbrace{000\ldots0}_{n-k \text{ puta}}$$

by preserving the order among same digits. Lets note that each palindrome consisting of same digits is in the initial code if and only if it is in the transformed code. The transformed code doesn't have a palindrome not consisting of same digits and thus the transformed code has less or equal palindromes than the initial one.

Thus we conclude that it is enough to consider only the codes starting with k digits one and ending in $n-k$ zeros, for some $k \in \{0, 1, \ldots n\}$.

Let us fix a $k \in \{0, 1, \dots n\}$. The code consisting of k ones and $n-k$ zeros has $2^k - 1 + 2^{n-k} - 1 = 2^k + 2^{n-k} - 2$ palindromes. We now seek k which minimizes this expression.

If n is even ($n = 2m$), by the AM-GM inequality $2^k + 2^{n-k} \geqslant 2 \cdot \sqrt{2^{k+n-k}} = 2^m + 2^m \implies$ the least possible number of palindromes in the code with $2m$ digits is $2^m + 2^m - 2 = 2^{m+1} - 2$, and this number is clearly attained for the code with m digits one and ending in m digits zero.

If n is odd ($n = 2m+1$) we have the following inequality for each $k \in \{0, 1, \dots m-1\}$:

$$2^k + 2^{n-k} > 2^{k+1} + 2^{n-k-1} \ (\iff 2^{n-k-1} > 2^k)$$

From this we also get $2^k + 2^{n-k-1} < 2^{k-1} + 2^{n-k+1}$ for all $k \in \{m+1, m+2, \dots 2m+1\}$. So:

$$2^0 + 2^n > 2^1 + 2^{n-1} > \dots > 2^m + 2^{m+1} = 2^{m+1} + 2^m < 2^{m+2} + 2^{m-1} < \dots < 2^n + 2^0$$

Now it is clear that the least number of palindromes in the code with $2m+1$ digits is $2^m + 2^{m+1} - 2$ and this number is attained by the code of m digits one and $m+1$ digits zero.

b) For $n = 1$ we clearly see that the answer is 1. From now on we assume $n \geqslant 2$.

As well for simplicity of the write-up we will not consider the one-digit palindromes as nice as we know that each code of n digits consists of n one-digit palindromes, each of which is nice. So we will find the smallest possible number of multi-digit nice palindromes and we will add n to this number to get the desired solution.

As a last remark: in this part of the solution for brevity we will denote as palindromes only those that are nice by the definitions in the problem statement.

Code consisting of n digits 1 contains one n-digit palindrome, two $(n-1)$-digit palindromes, ..., $n-2$ three digit palindromes and $n-1$ two digit palindromes. After summing up we get that this code has $\frac{n(n-1)}{2}$ palindromes. Analogously the code consisting of n digits 0 contains the same number of palindromes.

We now consider the code which contains at least one digit one and at least one digit zero. Then each digit 1 except the rightmost one is the start of at least one palindrome (the sequence of digits starting with it and ending in the first digit one to the right of it is of the form $100\dots01$ and is thus a palindrome). Analogously we conclude that each digit 0 apart from the rightmost one is a start of at least one palindrome. As we have at least one digit 1 and one digit 0 we conclude that each code consists of at least $n-2$ palindromes (where we have deducted 2 for the rightmost digit 1 and 0).

By induction on n we will show that for each $n \in \mathbb{N}, n \geqslant 2$ we can find a code with exactly $n-2$ palindromes. We can note that for $n = 2, 3, 4$ this is possible as the examples are (10), $(101), (1101)$. Now let's assume that the induction claim holds for some $n \in \mathbb{N}, n \geqslant 4$, and let $(x_1 \dots x_n)$ be a code with exactly $n-2$ palindromes.

That code is certainly not $(011\dots1)$ or $(100\dots0)$ (similarly as in the case with all digits equal we conclude that these codes have $\frac{(n-1)(n-2)}{2} > n-2$ palindromes).

We now that each of the digits one/zero apart from the rightmost ones is the start of at least one palindrome. In order for total number of palindromes to be $n-2$ all such digits are starts of exactly one palindrome. As $(x_1 \dots x_n) \neq (011\dots1)$ and $(x_1 \dots x_n) \neq (100\dots0)$, digit x_1 is not the rightmost digit one/zero $\implies x_1$ is the start of exactly one palindrome.

We now show that we can choose a digit x_0 such that $(x_0x_1x_2\dots x_n)$ contains exactly $n-1$ palindromes. As there are $n-2$ palindromes $(x_1x_2\dots x_n)$ we need to show that we can choose x_0 such that x_0 is a start of exactly one palindrome in $(x_0x_1\dots x_n)$. We know that x_0 is a start of at least one palindrome so we actually only have to show it is a start of at most one palindromes.

Let's consider to which palindromes can x_0 be a start:

- (x_0x_1) is a palindrome $\iff x_0 = x_1$
- $(x_0x_1x_2)$ is a palindrome $\iff x_0 = x_2$
- $(x_0x_1x_2\dots x_kx_{k+1})$ is a palindrome, for some $k \in \{2, 3, 4, \dots, n-1\} \iff x_0 = x_{k+1}$ and $(x_1x_2\dots x_k)$ is a palindrome

As there is exactly one palindrome for which x_1 is the start we conclude there is at most one palindrome such that x_0 is its start and it has the form as in the third case above. Thus there are at most three palindromes to which x_0 can be the first digit as we have two options for the choice of $x_0 \in \{0, 1\}$. Thus, by The Pigeonhole Principle we can choose a digit such that x_0 is a start of at most one palindrome, as desired.

Now using this and the remarks given before we have shown that the smallest possible number of nice palindromes with n digits is 1 (for $n = 1$) and $2n-2$ (for $n \geqslant 2$).

Problem 4. Given a triangle ABC let D, E, F be orthogonal projections from A, B, C to the opposite sides respectively. Let X, Y, Z denote midpoints of AD, BE, CF respectively. Prove that perpendiculars from D to YZ, from E to XZ and from F to XY are concurrent.

(Matija Bucić)

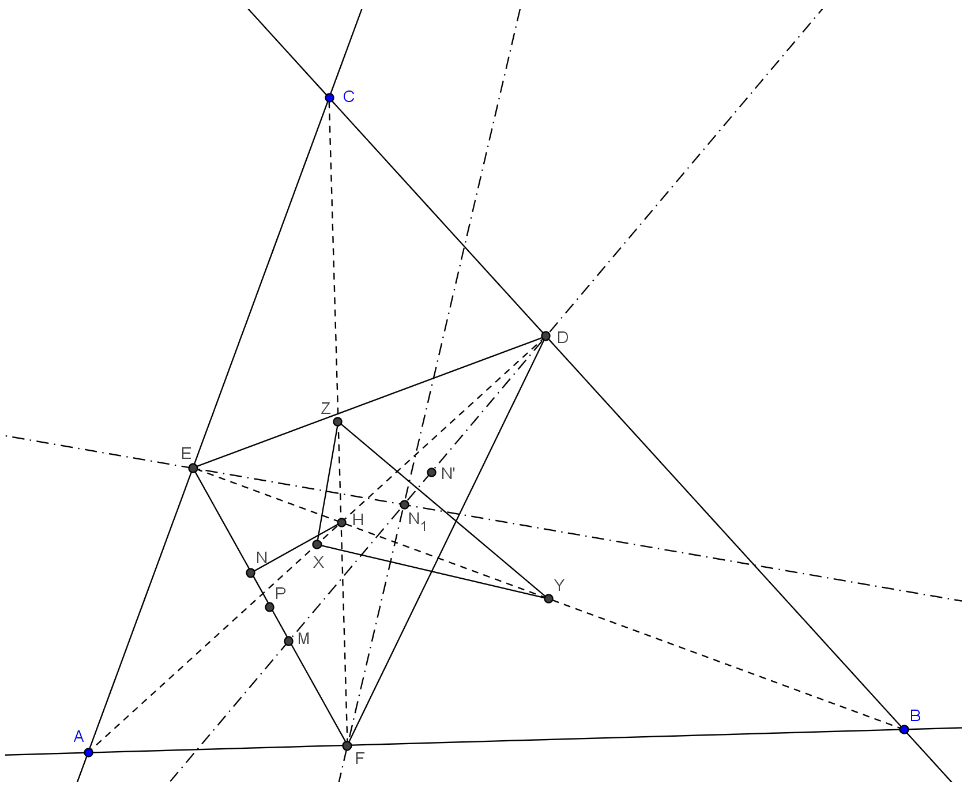
First solution. Let H be the orthocenter of the triangle ABC. We denote the midpoint of EF as P. As PZ is a midline of the triangle CEF we have $PZ||AC$, and as YH is perpendicular to AC, we get that YH is perpendicular to PZ. Analogously we conclude that the line ZH is perpendicular to PY, so H has to be the orthocenter of the triangle PYZ. From this we can deduce that the line PH is perpendicular to YZ, and thus PH is parallel to the line perpendicular to YZ which passes through D.

Now denote as N the tangency point of the incircle of the triangle DEF with its side EF. Let N' be the point symmetric to N with respect to H and let M be the tangency point of the D-excircle of the triangle DEF with the side EF. As P is the midpoint of $\overline{NM}$ and as is H the midpoint of NN', we have that PH is parallel to $N'M$. As we know that M is the map of the point N' under the homothety with centre D which maps the incircle to excircle of the triangle DEF, we can conclude that D, N' and M are collinear.

We can now conclude that the line perpendicular to YZ passing through D is parallel to PH while this line is parallel to $N'M$. As D lies on $N'M$ we conclude that DM is the line through D perpendicular to YZ.

Analogously we can conclude that perpendiculars from E to XZ and from F to XY are lines joining vertices with the corresponding excircle tangency point of the triangle DEF. Using the Ceva's Theorem gives us the result.

Remark: The intersection of the lines connecting the vertices of the triangle respective tangency points intersect in the point which is called *Nagel's* point of the triangle (so we have proved that the three lines in the problem intersect in the Nagel's point of the triangle DEF).



Second solution. By applying The Carnot's Theorem to the triangle XYZ and points D, E, F, three lines in the problem are concurrent if and only if:

$$FX^2 - FY^2 + DY^2 - DZ^2 + EZ^2 - EX^2 = 0 \tag{1}$$

In the triangle AFD and EFB lines $\overline{FX}$ and $\overline{FY}$ are medians, so

$$FX^2 = \frac{1}{4}(2AF^2 + 2FD^2 - AD^2)$$
$$FY^2 = \frac{1}{4}(2FB^2 + FE^2 - EB^2).$$

Noting that the other sides on the LHS of (1) are medians in the respective triangles we deduce:

$$\begin{gathered}FX^2 - FY^2 + DY^2 - DZ^2 + EZ^2 - EX^2 = \\ \frac{1}{4}[(2AF^2 + \cancel{2FD^2} - \bcancel{AD^2}) - (2FB^2 + \cancel{2FE^2} - \bcancel{EB^2}) + \\ +(2DB^2 + \cancel{2DE^2} - \bcancel{EB^2}) - (2DC^2 + \cancel{2DF^2} - \bcancel{CF^2}) + \\ +(2EC^2 + \cancel{2EF^2} - \bcancel{CF^2}) - (2EA^2 + \cancel{2ED^2} - \bcancel{AD^2})] = \\ \frac{1}{2}(AF^2 - FB^2 + DB^2 - DC^2 + EC^2 - EA^2).\end{gathered}$$

From right-angled triangles AFC and FBC we get:

$$AF^2 - FB^2 = (AC^2 - FC^2) - (BC^2 - FC^2) = AC^2 - BC^2.$$

Applying this analogously to triangles AEB, EBC, ADC, ADB we get:

$$\begin{gathered}FX^2 - FY^2 + DY^2 - DZ^2 + EZ^2 - EX^2 = \\ \frac{1}{2}(AF^2 - FB^2 + DB^2 - DC^2 + EC^2 - EA^2) = \\ \frac{1}{2}(AC^2 - BC^2 + AB^2 - AC^2 + BC^2 - AB^2) = 0,\end{gathered}$$

Q.E.D.