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ABOUT THE TYPES OF HOMOGENOUS LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER AND THEIR SOLUTIONS

Lazo Dimov

Abstract In the paper, types of linear differential equations of second order are defined in the sense of the known classification of the types of linear partial differential equations of second order. Then the shapes of their solutions are examined and some classes differential equations are solved.

1. PART 1

For the linear partial differential equations of second order

$$A\frac{\partial^2 z}{\partial x^2}+2B\frac{\partial^2 z}{\partial x\partial y}+C\frac{\partial^2 z}{\partial y^2}+D\frac{\partial z}{\partial x}+E\frac{\partial z}{\partial y}+Fz+G=0,$$

where A,B,C,D,E,F and G are real functions from the two variables x and y, depending on the value of the determinant

$$\Delta=\begin{vmatrix}A & B\\ B & C\end{vmatrix},$$

we have the following classification:

1. If $\Delta<0$, it is a PDE of hyperbolic type.
2. If $\Delta=0$, it is a PDE of parabolic type.
3. If $\Delta>0$, it is a PDE of elliptic type.

Here, we will make a similar classification for the homogenous linear differential equations of second order. It is well known that the equation

$$y''+Ay=0 \tag{1}$$

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where $A = const$ (equation 2.9 on page 365 in [2]), has a solution which can be written with the formula

$$y(x) = c_1 \sin\sqrt{A}x + c_2 \cos\sqrt{A}x, \quad \text{for} \quad A > 0$$
$$y(x) = c_1 x + c_2, \quad \text{for} \quad A = 0 \tag{2}$$
$$y(x) = c_1 \,\text{sh}\sqrt{-A}x + c_2 \,\text{ch}\sqrt{-A}x, \quad \text{for} \quad A < 0.$$

By analogy with the classification of the linear PDE of second order that we made, we can make the following classification for the equation (1) and its solution (2):

1. If $A < 0$, it is an equation of hyperbolic type.
2. If $A = 0$, it is an equation of parabolic type.
3. If $A > 0$, it is an equation of elliptic type.

If $A = a(x)$ is a given function, then we can make the following classification of the equation (1), which now has the form

$$y'' + a(x)y = 0 \tag{3}$$

and its solution as well, i.e.:

1. An equation of hyperbolic type, if $a(x) < 0$.
2. An equation of parabolic type, if $a(x) = 0$.
3. An equation of elliptic type, if $a(x) > 0$.

For the solution of the equation (3) for every type, we examine the following formulas:

$$y(x) = c_1 \sin_{a(x)} x + c_2 \cos_{a(x)} x, \quad \text{for} \quad a(x) > 0$$
$$y(x) = c_1 x + c_2, \quad \text{for} \quad a(x) = 0 \tag{4}$$
$$y(x) = c_1 \,\text{sh}_{a(x)} x + c_2 \,\text{ch}_{a(x)} x, \quad \text{for} \quad a(x) < 0.$$

Here, the functions $\sin_{a(x)} x, \cos_{a(x)} x, \text{sh}_{a(x)} x, \text{ch}_{a(x)} x$ are defined and determined in [3].

It is well known that if in the differential equation

$$y'' + f(x)y' + g(x)y = 0 \tag{5}$$

we introduce a new function given with

$$y(x) = e^{-\frac{1}{2}\int f(x)dx} z(x) \tag{6}$$

it will transform in the differential equation

$$z'' + a(x)z = 0 \tag{7}$$

known as canonical equation, where

$$a(x) = g(x) - \frac{1}{2} f'(x) - \frac{1}{4} f^2(x). \tag{8}$$

Now, according to the previous classification we made, we get the following classification for the equation (5):

1. The equation (5) is of hyperbolic type, if $a(x) < 0$.
2. The equation (5) is of parabolic type, if $a(x) = 0$.
3. The equation (5) is of elliptic type, if $a(x) > 0$.

Here, the solution of the equation (5), we get with the formulas:

$$y(x) = e^{-\frac{1}{2}\int f(x)dx}[c_1 \sin_{a(x)} x + c_2 \cos_{a(x)} x], \quad \text{for} \quad a(x) > 0$$

$$y(x) = e^{-\frac{1}{2}\int f(x)dx}[c_1 x + c_2], \quad \text{for} \quad a(x) = 0 \tag{9}$$

$$y(x) = e^{-\frac{1}{2}\int f(x)dx}[c_1 \,\mathrm{sh}_{a(x)} x + c_2 \,\mathrm{ch}_{a(x)} x], \quad \text{for} \quad a(x) < 0 .$$

Example. The differential equation

$$y'' + 2(x+k)y' + (x^2 + 2kx + 2k^2 - k + 1)y = 0$$

where

$$f(x) = 2(x+k), \quad g(x) = x^2 + 2kx + 2k^2 - k + 1,$$

and $k = const$ has a canonical equation

$$z'' + (k^2 - k)z = 0,$$

so, according to the formulas (2) has a solution

$$z(x) = c_1 \sin\sqrt{k^2 - k}x + c_2 \cos\sqrt{k^2 - k}x, \quad \text{if} \quad k \in (-\infty, 0) \cup (1, \infty)$$

$$z(x) = c_1 x + c_2, \quad \text{if} \quad k = 0 \text{ and } k = 1$$

$$z(x) = c_1 e^{-\sqrt{k-k^2}x} + c_2 e^{\sqrt{k-k^2}x}, \quad \text{if} \quad k \in (0,1).$$

Now, the given equation according to the formulas (9) has a solution:

$$y(x) = e^{-\frac{x^2}{2}+kx}[c_1 \sin\sqrt{k^2 - k}x + c_2 \cos\sqrt{k^2 - k}x], \quad \text{if} \quad k \in (-\infty, 0) \cup (1, \infty)$$

$$y(x) = e^{-\frac{x^2}{2}}[c_1 x + c_2], \quad \text{if} \quad k = 0$$

$$y(x) = e^{-\frac{x^2}{2}+x}[c_1 x + c_2], \quad \text{if} \quad k = 1$$

$$y(x) = e^{-\frac{x^2}{2}+kx}[c_1 e^{-\sqrt{k-k^2}x} + c_2 e^{\sqrt{k-k^2}x}], \quad \text{if} \quad k \in (0,1).$$

2. PART 2

If we introduce a new independent variable in the equation (5) with the relation $x = x(t)$, in order to get the equation (5) shaped as (1), we get that the functions that appear in the equation, satisfy the condition

$$g'(x) + 2f(x)g(x) = 0 \tag{10}$$

and we get the new variable from the relation

$$x'(t)^2 = \frac{A}{g(x)}. \tag{11}$$

From (11) we have the following:

1. for $g(x) > 0$ we have $A > 0$, so the equation becomes an equation from elliptic type, where the connection between the new and the old variable is $\sqrt{A}t = \int \sqrt{g(x)}\, dx.$

2. for $g(x) < 0$ we have $A < 0$, so the equation becomes an equation from hyperbolic type, where the connection between the new and the old variable is $\sqrt{-A}t = \int \sqrt{-g(x)}\, dx.$

3. if $g(x) = 0$, we can say that the equation is from parabolic type. Actually, the equation is

$$y'' + f(x)y' = 0$$

and its solution is given with the formula

$$y(x) = c_1 \int e^{-\int f(x)dx} dx + c_2.$$

Now, for the general solution of this kind of equation according to the formula (2), we get the following formula:

$$\begin{aligned} y(x) &= c_1 e^{-\int \sqrt{-g(x)}dx} + c_2 e^{\int \sqrt{-g(x)}dx}, \quad \text{for } g(x) > 0 \\ y(x) &= c_1 \int e^{-\int f(x)dx} dx + c_2, \quad \text{for } g(x) = 0 \\ y(x) &= c_1 \sin\left(\int \sqrt{g(x)}dx\right) + c_2 \cos\left(\int \sqrt{g(x)}dx\right), \quad \text{for } g(x) < 0. \end{aligned} \tag{12}$$

Example. In the differential equation

$$y'' + \frac{1}{2x} y' + \frac{k}{x} y = 0,$$

where the functions $f(x) = \frac{1}{2x}$ and $g(x) = \frac{k}{x}$ satisfy the condition (10). So, according to the formulas (12), its solution is

$$y(x)=c_1e^{-2\sqrt{-kx}}+c_2e^{2\sqrt{-kx}}, \quad \text{for} \quad kx>0$$
$$y(x)=c_1\sqrt{x}+c_2, \quad \text{for} \quad k=0 \qquad (12')$$
$$y(x)=c_1\sin(2\sqrt{kx})+c_2\cos(2\sqrt{kx}), \quad \text{for} \quad kx<0.$$

3. PART 3

Here we will explore the differential equation of second order with positive constant coefficients

$$y''(t)+2ay'(t)+b^2y(t)=0, \qquad (13)$$

(in [5] it is explored with dimensional constants). The characteristic equation of the equation (13) is

$$r^2+2ar+b^2=0.$$

Its solutions are $$r_{1/2}=-a\pm\sqrt{a^2-b^2}.$$

It is useful to write them in the following shape

$$r_{1/2}=(-\tfrac{a}{b}\pm\sqrt{(\tfrac{a}{b})^2-1})b=(-k\pm\sqrt{l})b, \qquad k=\tfrac{a}{b},\ l=k^2-1.$$

Now, depending on the value of l, we make the following classification of the equation (13):

1. for $l<0$ i.e. $k<1$, the equation is an equation from elliptic type,
2. for $l=0$ i.e. $k=1$, the equation is an equation from parabolic type,
3. for $l>0$ i.e. $k>1$, the equation is an equation from hyperbolic type.

Its solution is given with the formulas:

$$y(x)=e^{-btk}(c_1\cos\sqrt{-l}bt+c_2\sin\sqrt{-l}bt), \quad \text{for } l<0, \text{ i.e. } k<1$$
$$y(x)=e^{-bt}(c_1+c_2t), \quad \text{for} \quad l=0, \text{ i.e. } k=1 \qquad (14)$$
$$y(x)=c_1e^{btm}+c_2e^{btn}, \text{ for } l>0, \text{ i.e. } k>1, \text{ where } m=-k-\sqrt{l}, n=-k+\sqrt{l}.$$

Now, if in the differential equation (5) we introduce new independent variable with the relation $x=x(t)$, in order to get the equation (5) shaped as the differential equation (13), we get that the following conditions have to be satisfied

$$f(x)x'(t)-\frac{x''(t)}{x'(t)}=2a \qquad (15)$$

$$g(x)x'(t)^2=b^2. \qquad (16)$$

Eliminating t from (15) and (16) we get

$$\frac{2g(x)f(x)+g'(x)}{4g(x)\sqrt{g(x)}}=\frac{a}{b}=k. \qquad (17)$$

Now, if we find the derivative of (17) we get the functional connection:

$$2f(x)g(x)g'(x)-4g^2(x)f'(x)-2g(x)g''(x)+3g'(x)^2=0 \qquad (18)$$

which in fact is the condition that has to satisfy the functions $f(x)$ and $g(x)$ in order the equation (5) to be able to transform into (13).

From (16) we come to the connection between the old and the new independent variable:

$$bt=\int\sqrt{g(x)}dx. \qquad (19)$$

So, we have proven the following theorem:

Theorem. The differential equation (5) can be transformed into a differential equation of type (13) if the functions $f(x)$ and $g(x)$ satisfy the condition (18) and the new independent variable is given with the relation (19). Here, we get the general solution according to the formulas (14) in which we substitute bt with (19) and the ratio $\frac{a}{b}=k$ is given with the formula (17).

For the general solution we have the formulas:

$$y=e^{-k\int\sqrt{g(x)}dx}\left(c_1\cos l\int\sqrt{g(x)}dx+c_2\sin l\int\sqrt{g(x)}dx\right), \text{ for } 0<k<1,\ l=\sqrt{1-k^2}.$$

$$y=e^{-\int\sqrt{g(x)}dx}(c_1+c_2\int\sqrt{g(x)}dx), \quad \text{for } k=1,$$

$$y=c_1e^{m\int\sqrt{g(x)}dx}+c_2e^{n\int\sqrt{g(x)}dx}, \text{ for } k>1, \text{ where } m=-k-\sqrt{-l}, n=-k+\sqrt{-l}.$$

Example 1. In the differential equation

$$y''+(c\sqrt{x}-\tfrac{1}{2x})y'+xy=0,$$

the functions $f(x)=c\sqrt{x}-\frac{1}{2x}$ and $g(x)=x$ satisfy the condition (18), for k from (17) we get $k=\frac{c}{2}$. So, according to the formula for the substitute, we have

$$\int\sqrt{g(x)}dx=\int\sqrt{x}dx=\tfrac{2}{3}x^{\frac{3}{2}}.$$

Depending on the value of c we have the following forms for the solution of the differential equation

$$y=e^{-\frac{c}{3}x^{\frac{3}{2}}}(c_1\cos\tfrac{1}{3}\sqrt{4-c^2}x^{\frac{3}{2}}+c_2\sin\tfrac{1}{3}\sqrt{4-c^2}x^{\frac{3}{2}}), \quad \text{for } 0<c<2.$$

$$y=(c_1+c_2x^{\frac{3}{2}})e^{-\frac{2}{3}x^{\frac{3}{2}}}, \quad \text{for} \quad c=2,$$

$$y=c_1e^{-\frac{c+\sqrt{c^2-4}}{3}x^{\frac{3}{2}}}+c_2e^{\frac{c+\sqrt{c^2-4}}{3}x^{\frac{3}{2}}}, \quad \text{for} \quad c>2.$$

Example 2. The Euler differential equation

$$x^2y''+cxy'+y=0,$$

written in shape (5) is

$$y''+\frac{c}{x}y'+\frac{1}{x^2}y=0\,.$$

The functions $f(x)=\frac{c}{x}$ and $g(x)=\frac{1}{x^2}$ satisfy the condition (18) and for k from (17) we get $k=\frac{c-1}{2}$. So, according to the formula for the substitute, we have

$$\int\sqrt{g(x)}dx=\int\frac{1}{x}dx=\ln x.$$

Depending on the value of c we have the following forms for the solution of the differential equation

$$y=e^{-\frac{c-1}{3}}(c_1\cos\tfrac{1}{2}\sqrt{3+2c-c^2}\ln x+c_2\sin\tfrac{1}{2}\sqrt{3+2c-c^2}\ln x), \quad \text{for} \quad 1<c<3,$$

$$y=(c_1+c_2\ln x)\frac{1}{x}, \quad \text{for} \quad c=3,$$

$$y=c_1e^{\frac{1-c-\sqrt{c^2-2c-3}}{2}}+c_2e^{\frac{1-c+\sqrt{c^2-2c-3}}{2}}, \quad \text{for} \quad c>3.$$

Note. In [2] on page 383, ex. 2.75 is proven that the differential equation (5) with similar substitution can be transformed into the differential equation with constant coefficients

$$y''+ay'+y=0,$$

which is a special case of the equation (13).

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SEQUENTIALLY CONVERGENT MAPPINGS AND COMMON FIXED POINTS OF MAPPINGS IN 2-BANACH SPACES

Martin Lukarevski[1], Samoil Malčeski[2]

Abstract. In the past few years, the classical results about the theory of fixed point are transmitted in 2-Banach spaces, defined by A. White (see [3] and [8]). Several generalizations of Kannan, Chatterjea and Koparde-Waghmode theorems are given in [1], [4], [5] and [7]. In this paper, several generalizations of already known theorems about common fixed points of mappings in 2-Banach spaces, are proven, by using the sequentially convergent mappings.

1. INTRODUCTION

In 1968 White ([3]) introduces 2-Banach spaces. 2-Banach spaces are being studied by several authors, and certain results can be seen in [8]. Further, analogously as in normed space P. K. Hatikrishnan and K. T. Ravindran in [6] are introducing the term contraction mapping to 2-normed space as follows.

Definition 1 ([6]). Let $(L,\|\cdot,\cdot\|)$ be a real vector 2-normed space. The mapping $S: L \to L$ is contraction if there is $\lambda \in [0,1)$ such that

$$\| Sx - Sy, z \| \le \lambda \| x - y, z \|, \text{ for all } x, y, z \in L .$$

Regarding contraction mapping Hatikrishnan and Ravindran in [6] proved that contraction mapping has a unique fixed point in closed and restricted subset of 2-Banach space. Further, in [1], [4], [5] and [7] are proven more results related to fixed points of contraction mapping of 2-Banach spaces, and in [7] are proven several results for common fixed points of contraction mapping defined on the same 2-Banach space.

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In our further considerations, we will give some generalizations of the above results for common fixed points of mapping defined on the same 2-Banach space. Thus, the mentioned generalizations we will do with the help of so-called sequentially convergent mappings which are defined as follows.

Definition 2. Let $(L,\|\cdot,\cdot\|)$ be a 2-normed space. A mapping $T:L\to L$ is said to be sequentially convergent if, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent then $\{y_n\}$ also is convergent.

2. COMMON FIXED POINTS OF MAPPING OF THE KANNAN TYPE

Theorem 1. Let $(L,\|\cdot,\cdot\|)$ be a 2- Banach space, $S_1,S_2:L\to L$ and mapping $T:L\to L$ is continuous, injection and sequentially convergent. If $\alpha>0$, $\gamma\geq 0$ are such that $2\alpha+\gamma<1$ and

$$\|TS_1x-TS_2y,z\|\leq\alpha(\|Tx-TS_1x,z\|+\|Ty-TS_2y,z\|)+\gamma\|Tx-Ty,z\|, \qquad (1)$$

for each $x,y,z\in L$, then S_1 and S_2 have a unique common fixed point $z\in L$.

Proof. Let x_0 be an arbitrary point of L and let the sequence $\{x_n\}$ be defined with $x_{2n+1}=S_1x_{2n}$, $x_{2n+2}=S_2x_{2n+1}$, for $n=0,1,2,...$. If there is $n\geq 0$ such that $x_n=x_{n+1}=x_{n+2}$, then it is easy to prove that $u=x_n$ is a common fixed point for S_1 and S_2. Therefore, let's assume that there do not exist three different consecutive equal members of the sequence $\{x_n\}$. So, using inequalities (1), it is easy to prove that for each $n\geq 1$ and for each $z\in L$ the following holds true

$$\|Tx_{2n+1}-Tx_{2n},z\|\leq\alpha(\|Tx_{2n+1}-Tx_{2n},z\|+\|Tx_{2n}-Tx_{2n-1},z\|)+\gamma\|Tx_{2n}-Tx_{2n-1},z\|$$

and

$$\|Tx_{2n-1}-Tx_{2n},z\|\leq\alpha(\|Tx_{2n-2}-Tx_{2n-1},z\|+\|Tx_{2n-1}-Tx_{2n},z\|)$$
$$+\gamma\|Tx_{2n-2}-Tx_{2n-1},z\|,$$

from which it follows that

$$\|Tx_{n+1}-Tx_n,z\|\leq\lambda\|Tx_n-Tx_{n-1},z\|, \qquad (2)$$

for each $n=0,1,2,...$, where $\lambda=\frac{\alpha+\gamma}{1-\alpha}<1$. Now from inequality (2) it follows that

$$\|Tx_{n+1}-Tx_n,z\|\leq\lambda^n\|Tx_1-Tx_0,z\|, \qquad (3)$$

for each $z \in L$ and for each $n = 0,1,2,\ldots$. But, then from inequality (3) follows that for each $m,n \in \mathbf{N}$, $n > m$ and for each $z \in L$ the following holds true

$$\| Tx_n - Tx_m, z \| \le \frac{\lambda^m}{1-\lambda} \| Tx_1 - Tx_0, z \|,$$

which means that the sequence $\{Tx_n\}$ is Cauchy and because space L is 2-Banach we get that the sequence $\{Tx_n\}$ is convergent. Further, the mapping $T : L \to L$ is sequentially convergent and because the sequence $\{Tx_n\}$ is convergent, from definition 2 follows that the sequence $\{x_n\}$ is convergent, i.e. exists $u \in L$ such that $\lim_{n\to\infty} x_n = u$. Now from the continuity of T follows that $\lim_{n\to\infty} Tx_n = Tu$. Then, for each $z \in L$ the following holds true

$$\begin{aligned}
\| Tu - TS_1u, z \| &\le \| Tu - Tx_{2n+2}, z \| + \| Tx_{2n+2} - TS_1u, z \| \\
&= \| Tu - Tx_{2n+2}, z \| + \| TS_2x_{2n+1} - TS_1u, z \| \\
&\le \| Tu - Tx_{2n+2}, z \| + \alpha(\| Tu - TS_1u, z \| + \| Tx_{2n+1} - TS_2x_{2n+1}, z \|) \\
&\quad + \gamma \| Tu - Tx_{2n+1}, z \| \\
&\le \| Tu - Tx_{2n+2}, z \| + \alpha(\| Tu - TS_1u, z \| + \| Tx_{2n+1} - Tx_{2n+2}, z \|) \\
&\quad + \gamma \| Tu - Tx_{2n+1}, z \|.
\end{aligned}$$

If in the last inequality we take that $n \to \infty$, for each $z \in L$ the following holds true

$$\| Tu - TS_1u, z \| \le \alpha \| Tu - TS_1u, z \|,$$

and since $\alpha < 1$, we conclude that $\| TS_1u - Tu, z \| = 0$, for each $z \in L$, i.e. $TS_1u = Tu$. But, T is injection, so $S_1u = u$, i.e. u is fixed point on S_1. Analogously can be proved that u is fixed point of S_2. Let $v \in L$ is another fixed point of S_2, i.e. $S_2v = v$. Then, for each $z \in L$ the following holds true

$$\begin{aligned}
\| Tu - Tv, z \| &= \| TS_1u - TS_2v, z \| \\
&\le \alpha(\| Tu - TS_2v, z \| + \| Tv - TS_1u, z \|) + \gamma \| Tu - Tv, z \| \\
&= (2\alpha + \gamma) \| Tu - Tv, z \|,
\end{aligned}$$

and as $2\alpha + \beta < 1$ we get that for each $z \in L$ the following holds true $\| Tu - Tv, z \| = 0$, from which follows $Tu = Tv$. But, T is injection, so $u = v$. ■

Corollary 1. Let $(L, \| \cdot, \cdot \|)$ be a 2- Banach space, $S_1, S_2 : L \to L$ and mapping $T : L \to L$ is continuous, injection and sequentially convergent. If $\alpha > 0$, $\gamma \ge 0$ are such that $2\alpha + \gamma < 1$ and

$$\| TS_1x - TS_2y, z \| \leq \alpha \frac{\|Tx-TS_1x,z\|^2 + \|Ty-TS_2y,z\|^2}{\|Tx-TS_1x,z\| + \|Ty-TS_2y,z\|} + \gamma \| Tx - Ty, z \|,$$

for each $x, y, z \in L$, $z \neq 0$, then S_1 and S_2 have a unique common fixed point $z \in L$.

Proof. From inequality of condition follows inequality (1). Now the assertion follows from Theorem 1. ■

Corollary 2. Let $(L, \|\cdot,\cdot\|)$ be a 2- Banach space, $S_1, S_2 : L \to L$ and mapping $T : L \to L$ is continuous, injection and sequentially convergent. If $0 < \lambda < 1$ and

$$\| TS_1x - TS_2y, z \| \leq \lambda \cdot \sqrt[3]{\| Tx - TS_1x, z \| \cdot \| Ty - TS_2y, z \| \cdot \| Tx - Ty, z \|},$$

for each $x, y, z \in L$, then S_1 and S_2 have a unique common fixed point $z \in L$.

Proof. From the inequality between the arithmetic and geometric mean follows that

$$d(TS_1x, TS_2y) \leq \tfrac{\lambda}{3}(d(Tx, TS_1x) + d(Ty, TS_2y) + \beta d(Tx, Ty)).$$

Now the assertion follows from Theorem 1 for $\alpha = \gamma = \frac{\lambda}{3}$. ■

Corollary 3. Let $(L, \|\cdot,\cdot\|)$ be a 2- Banach space, $S_1^p, S_2^q : L \to L$, $p, q \in \mathbf{N}$ and mapping $T : L \to L$ is continuous, injection and sequentially convergent. If $\alpha > 0, \gamma \geq 0$ are such that $2\alpha + \gamma < 1$ and

$$\| TS_1^p x - TS_2^q y, z \| \leq \alpha(\| Tx - TS_1^p x, z \| + \| Ty - TS_2^q y, z \|) + \gamma \| Tx - Ty, z \|,$$

for each $x, y, z \in L$. Then S_1 and S_2 have a unique common fixed point $u \in L$.

Proof. From Theorem 1 follows that mappings S_1^p and S_2^q have a unique common fixed point $u \in L$. That means $S_1^p u = u$, so $S_1 u = S_1(S_1^p u) = S_1^p(S_1 u)$, and $S_1 u$ is fixed point of S_1^p. Analogously, we can prove that $S_2 u$ is fixed point of S_2^q. But, from the proof of Theorem 1 follows that mappings S_2^q and S_1^p have unique fixed point, so $u = S_2 u$ and $u = S_1 u$. According to that, $u \in L$ is a unique common fixed point of S_1 and S_2. Clearly, if $v \in L$ is another unique common fixed point of S_1 and S_2, then it is a common fixed point of S_1^p and S_2^q. But, S_1^p and S_2^q have a unique common fixed point, so $v = u$. ■

Remark 1. Mapping $T: L \to L$ defined by $Tx = x, x \in L$ is sequentially convergent. Therefore, if in theorem 1 and the corollaries 1, 2 and 3 we take that $Tx = x$ follows Theorem 4 and corollaries 6, 7 and 8, [7].

3. COMMON FIXED POINTS OF MAPPINGS OF CHATTERJEA TYPE

Theorem 2. Let $(L, \|\cdot,\cdot\|)$ be a 2- Banach space, $S_1, S_2 : L \to L$ and mapping $T: L \to L$ is continuous, injection and sequentially convergent. If $\alpha > 0$, $\gamma \geq 0$, are such that $2\alpha + \gamma < 1$ and

$$\| TS_1 x - TS_2 y, z \| \leq \alpha(\| Tx - TS_2 y, z \| + \| Ty - TS_1 x, z \|) + \gamma \| Tx - Ty, z \|, \qquad (4)$$

for each $x, y, z \in L$, then S_1 and S_2 have a unique common fixed point $u \in L$.

Proof. Let x_0 is arbitrary point from L and the sequence $\{x_n\}$ is defined with $x_{2n+1} = S_1 x_{2n}$, $x_{2n+2} = S_2 x_{2n+1}$, for $n = 0,1,2,...$. If there is $n \geq 0$ such that $x_n = x_{n+1} = x_{n+2}$, then $u = x_n$ is common fixed point of S_1 and S_2. Therefore, let's assume that there are three different consecutive equal members of the sequence $\{x_n\}$. Then, from nequality (4) follows that for every $z \in L$ and for every $n \geq 1$ the following holds true

$$\| Tx_{2n+1} - Tx_{2n}, z \| \leq \alpha(\| Tx_{2n-1} - Tx_{2n}, z \| + \| Tx_{2n} - Tx_{2n+1}, z \|) + \gamma \| Tx_{2n} - Tx_{2n-1}, z \|,$$

and

$$\| Tx_{2n-1} - Tx_{2n}, z \| \leq \alpha(\| Tx_{2n-2} - Tx_{2n-1}, z \| + \| Tx_{2n-1} - Tx_{2n}, z \|) + \gamma \| Tx_{2n-2} - Tx_{2n-1}, z \|,$$

so for each $z \in L$ and for each $n = 0,1,2,...$ the following holds true

$$\| Tx_{n+1} - Tx_n, z \| \leq \lambda \| Tx_n - Tx_{n-1}, z \|,$$

where $\lambda = \frac{\alpha+\gamma}{1-\alpha} < 1$. Then, for each $z \in L$ and for each $n = 0,1,2,...$ the following holds true

$$\| Tx_{n+1} - Tx_n, z \| \leq \lambda^n \| Tx_1 - Tx_0, z \|. \qquad (5)$$

Furthermore, using the inequality (5), in the same way as in the proof of Theorem 1 can be proved that the sequence $\{Tx_n\}$ is convergent, from where it follows that the sequence $\{x_n\}$ is convergent, i.e. there is $u \in L$ such that $\lim_{n\to\infty} x_n = u$ and $\lim_{n\to\infty} Tx_n = Tu$. We will prove that u is a fixed point of S_1. For each $z \in L$ we have

$$\begin{aligned}||Tu-TS_1u,z|| &\leq ||Tu-Tx_{2n+2},z||+||Tx_{2n+2}-TS_1u,z||\\ &=||Tu-Tx_{2n+2},z||+||TS_2x_{2n+1}-TS_1u,z||\\ &\leq ||Tu-Tx_{2n+2},z||+\alpha(||Tx_{2n+1}-TS_1u,z||+||Tu-TS_2x_{2n+1},z||)\\ &\quad +\gamma||Tu-Tx_{2n+1},z||\\ &\leq ||Tu-Tx_{2n+2},z||+\alpha(||Tx_{2n+1}-TS_1u,z||+||Tu-Tx_{2n+2},z||)\\ &\quad +\gamma||Tu-Tx_{2n+1},z||,\end{aligned}$$

and if in the last inequality we take $n\to\infty$ we get that for each $z\in L$ the following holds true $||Tu-TS_1u,z||\leq\alpha||Tu-TS_1u,z||$, and how $\alpha<1$, from the last inequality follows $||TS_1u-Tu,z||=0$, for each $z\in L$. Now, as in the proof of Theorem 1 we can conclude that u is fixed point of S_1. Analogously can be proved that u is fixed point of S_2. Let $v\in L$ is another fixed point of S_2, i.e. $S_2v=v$. For each $z\in L$ the following holds true

$$\begin{aligned}||Tu-Tv,z|| &=||TS_1u-TS_2v,z||\\ &\leq\alpha(||Tu-TS_2v,z||+||Tv-TS_1u,z||)+\gamma||Tu-Tv,z||\\ &=(2\alpha+\gamma)||Tu-Tv,z||.\end{aligned}$$

Since $2\alpha+\gamma<1$ from the last inequality it follows that for every $z\in L$ the following holds true $||Tu-Tv,z||=0$, from which follows that $Tu=Tv$. But, T is injection, so $u=v$. ■

Corollary 4. Let $(L,||\cdot,\cdot||)$ be a 2-Banach space, $S_1,S_2:L\to L$ and the mapping $T:L\to L$ is continuous, injection and sequentially convergent. If $\alpha>0$, $\gamma\geq 0$ are such that $2\alpha+\gamma<1$ and

$$||TS_1x-TS_2y,z||\leq\alpha\frac{||Tx-TS_2y,z||^2+||Ty-TS_1x,z||^2}{||Tx-TS_2y,z||+||Ty-TS_1x,z||}+\gamma||Tx-Ty,z||,$$

for each $x,y,z\in L$, $z\neq 0$, then S_1 and S_2 have a unique common fixed point $u\in L$.

Proof. From inequality of condition follows inequality (4). Now the assertion follows from Theorem 2.■

Corollary 5. Let $(L,||\cdot,\cdot||)$ be a 2-Banach space, $S_1,S_2:L\to L$ and mapping $T:L\to L$ is continuous, injection and sequentially convergent. If $0<\lambda<1$ and

$$||TS_1x-TS_2y,z||\leq\lambda\cdot\sqrt[3]{||Tx-TS_2y,z||\cdot||Ty-TS_1x,z||\cdot||Tx-Ty,z||},$$

for each $x,y,z\in L$, then S_1 and S_2 have a unique common fixed point $z\in L$.

Proof. From the inequality between the arithmetic and geometric mean follows that

$$d(TS_1x,TS_2y) \le \frac{\lambda}{3}(d(Tx,TS_2y)+d(Ty,TS_1x)+d(Tx,Ty)) .$$

Now the assertion follows from Theorem 2 for $\alpha=\gamma=\frac{\lambda}{3}$. ■

Corollary 6. Let $(L,\|\cdot,\cdot\|)$ be a 2-Banach space, $S_1^p, S_2^q : L \to L$, $p,q \in \mathbf{N}$ and mapping $T : L \to L$ is continuous, injection and sequentially convergent. If $\alpha > 0, \gamma \ge 0$ are such that $2\alpha+\gamma<1$ and

$$\| TS_1^p x - TS_2^q y, z \| \le \alpha(\| Tx - TS_2^q y, z \| + \| Ty - TS_1^p x, z \|) + \gamma \| Tx - Ty, z \| ,$$

for each $x, y, z \in L$. Then S_1 and S_2 have a unique common fixed point $u \in L$.
Proof. The proof is identical to the proof of the corollary 5. ■

Remark 2. The mapping $T : L \to L$ determined by $Tx = x,\ x \in L$ is sequentially convergent. Therefore, if in Theorem 2 and corollaries 4, 5 and 6 we take $Tx = x$, follows the correctness of Theorem 5 and corollaries 9, 10 и 11, [7].

4. COMMON FIXED POINTS OF MAPPINGS OF KOPARDE-WAGHMODE TYPE

Theorem 3. Let $(L,\|\cdot,\cdot\|)$ be a 2-Banach space, $S_1, S_2 : L \to L$ and mapping $T : L \to L$ is continuous, injection and sequentially convergent. If $\alpha > 0$, $\gamma \ge 0$, $2\alpha+\gamma<1$ and

$$\| TS_1x - TS_2y, z \|^2 \le \alpha(\| Tx - TS_1x, z \|^2 + \| Ty - TS_2y, z \|^2) + \gamma \| Tx - Ty, z \|^2 , \quad (6)$$

for each $x, y, z \in L$, then S_1 and S_2 have a unique common fixed point $u \in L$.

Proof. Let x_0 be an arbitrary point of L and let the sequence $\{x_n\}$ is defined with $x_{2n+1} = S_1x_{2n}$, $x_{2n+2} = S_2x_{2n+1}$, for $n = 0,1,2,...$. If there is an $n \ge 0$ such that $x_n = x_{n+1} = x_{n+2}$, then $u = x_n$ is a common fixed point for S_1 and S_2. Therefore, let's assume that there do not exist three consecutive equal members of the sequence $\{x_n\}$. Then, from inequality (6) follows that for each $n \ge 1$ and for each $z \in L$ the following holds true

$$\begin{aligned} \| Tx_{2n+1} - Tx_{2n}, z \|^2 \le \alpha(&\| Tx_{2n} - Tx_{2n+1}, z \|^2 + \| Tx_{2n-1} - Tx_{2n}, z \|^2) \\ &+ \gamma \| Tx_{2n} - Tx_{2n-1}, z \|^2, \end{aligned}$$

and

$$\| Tx_{2n-1} - Tx_{2n}, z \|^2 \le \alpha(\| Tx_{2n-2} - Tx_{2n-1}, z \|^2 + \| Tx_{2n-1} - Tx_{2n} \|^2) + \gamma \| Tx_{2n-2} - Tx_{2n-1}, z \|^2,$$

from which it follows that for each $n = 0,1,2,...$ and for each $z \in L$ the following holds true

$$\| Tx_{n+1} - Tx_n, z \| \le \lambda \| Tx_n - Tx_{n-1}, z \|, \tag{7}$$

where $\lambda = \sqrt{\frac{\alpha+\gamma}{1-\alpha}} < 1$. Now from inequality (7) follows

$$\| Tx_{n+1} - Tx_n, z \| \le \lambda^n \| Tx_1 - Tx_0, z \|, \tag{8}$$

for each $n = 0,1,2,...$ and for each $z \in L$. Furthermore, from inequality (8), in the same way as in the proof of Theorem 1 it follows that the sequence $\{Tx_n\}$ is convergent, and therefore the sequence $\{x_n\}$ is convergent also, i.e. exists $u \in X$ such that $\lim\limits_{n\to\infty} x_n = u$ and $\lim\limits_{n\to\infty} Tx_n = Tu$. We will prove that u is fixed point of S_1. We have

$$\begin{aligned}
\| Tu - TS_1 u, z \| &\le \| Tu - Tx_{2n+2}, z \| + \| Tx_{2n+2} - TS_1 u, z \| \\
&= \| Tu - Tx_{2n+2}, z \| + \| TS_1 u - TS_2 x_{2n+1}, z \| \\
&\le \| Tu - Tx_{2n+2}, z \| + \sqrt{\alpha(\| Tu - TS_1 u, z \|^2 + \| Tx_{2n+1} - TS_2 x_{2n+1}, z \|^2) + \gamma \| Tu - Tx_{2n+1}, z \|^2} \\
&= \| Tu - Tx_{2n+2}, z \| + \sqrt{\alpha(\| Tu - TS_1 u, z \|^2 + \| Tx_{2n+1} - Tx_{2n+2}, z \|^2 + \gamma \| Tu - Tx_{2n+1}, z \|^2}
\end{aligned}$$

for each $n \in \mathbf{N}$ and for each $z \in L$. If in the last inequality we take $n \to \infty$ we get that

$$\| Tu - TS_1 u, z \| \le \sqrt{\alpha} d \| Tu - TS_1 u, z \|,$$

for each $z \in L$ and how $\sqrt{\alpha} < 1$, it follows that $\| Tu - TS_1 u, z \| = 0$. Now, again as in the proof of Theorem 1 we conclude that u is fixed point of S_1. Analogously it can be proved that u is fixed point of S_2. Let $v \in L$ be another fixed point of S_2, i.e. $S_2 v = v$. Then, for each $z \in L$ the following holds true

$$\begin{aligned}
\| Tu - Tv, z \|^2 &= \| TS_1 u - TS_2 v, z \|^2 \\
&\le \alpha(\| Tu - TS_1 u, z \|^2 + \| Tv - TS_2 v, z \|^2) + \gamma \| Tu - Tv, z \|^2 \\
&= \gamma \| Tu - Tv, z \|^2,
\end{aligned}$$

and how $0 \le \beta < 1$ we get that $\| Tu - Tv, z \| = 0$, from where it follows that $Tu = Tv$. But, T is injection, so $u = v$. ■

Corollary 7. Let $(L,\|\cdot,\cdot\|)$ be a 2-Banach space, $S_1^p, S_2^q : L \to L$, $p,q \in \mathbf{N}$ and mapping $T : L \to L$ is continuous, injection and sequentially convergent. If $\alpha > 0, \gamma \ge 0$ are such that $2\alpha + \gamma < 1$ and

$$\| TS_1^p x - TS_2^q y, z \|^2 \le \alpha(\| Tx - TS_1^p x, z \|^2 + \| Ty - TS_2^q y, z \|^2) + \gamma \| Tx - Ty, z \|^2,$$

for each $x, y, z \in L$. Then S_1 and S_2 have a unique common fixed point $u \in L$.

Proof. The proof is identical to the proof of the corollary 6. ■

Remark 3. The mapping $T : L \to L$ determined by $Tx = x, x \in L$ is sequentially convergent. Therefore, if in Theorem 3 and corollary 7 we take $Tx = x$, it follows the correctness of Theorem 6 and corollary 12, [7].

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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ABOUT A TWIN SOLUTION OF THE VEKUA EQUATION

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Abstract. In the paper main object of research is the Vekua equation. Two types of functions are found that are strongly connected to each other because it will be proven that if one of them is a solution of the Vekua equation, so will be the other one with a corresponding condition. Three different cases are considered.

1. INTRODUCTION

G. V. Kolosov in 1909 [1], when he was solving a problem from the theory of elasticity, introduced the expressions

$$\frac{1}{2}[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+i(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y})=\frac{\hat{d}W}{dz} \qquad \text{and}$$

$$\frac{1}{2}[\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}+i(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y})=\frac{\hat{d}W}{d\overline{z}}$$

known as operatory derivatives of a complex function $W=W(z)=u(x,y)$ $+iv(x,y)$ from a complex variable $z=x+iy$ and $\overline{z}=x-iy$, respectively. The operator rules for these derivatives are given in the monograph of Г. Н. Положий [2] (pages 18-31). In the mentioned monograph, are also defined the so called operatory integrals

$$\hat{\int} f(z)dz \quad \text{and} \quad \hat{\int} f(z)d\overline{z}$$

by $z=x+iy$ and $\overline{z}=x-iy$, respectively, from the complex function $f=f(z)$ in the area $D\subseteq\mathbb{C}$, where their operatory rules are proven as well, page 32 - 41.

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2. FORMULATION OF THE PROBLEM

Main object of this research is the Vekua equation

$$\frac{\hat{d}W}{d\bar{z}} = A\overline{W} + BW + F \qquad (1)$$

where the functions $A = A(z)$, $B = B(z)$, $F = F(z)$ are arbitrary functions from complex variable without any limitation or condition that they have to fulfill.

Because in general case there is no method for finding its general solution, we explore the idea to find some solution of the Vekua equation (1) in the following form:

$$W = W(\varphi(\bar{z}), \psi(z)) \qquad (2)$$

where $\varphi = \varphi(\bar{z})$ is antyanalytic function and $\psi = \psi(z)$ is analytic function.

3. MAIN RESULT

Case 1. Let $W = \frac{\varphi(\bar{z})}{\psi(z)}$ be a solution of the equation (1) (and is from the form (2)), i.e. it is a ratio from one antyanalytic and one analytic function. That means that this function satisfies the equation (1), so if we find the operator derivative by $\bar{z}$ from W and replace it in (1) we get:

$$\frac{\hat{d}W}{d\bar{z}} = \frac{\hat{d}}{d\bar{z}}\left(\frac{\varphi(\bar{z})}{\psi(z)}\right) = \frac{1}{\psi(z)} \cdot \frac{\hat{d}\varphi(\bar{z})}{d\bar{z}}$$

$$\frac{1}{\psi(z)} \cdot \frac{\hat{d}\varphi(\bar{z})}{d\bar{z}} = A\overline{\left(\frac{\varphi(\bar{z})}{\psi(z)}\right)} + B\left(\frac{\varphi(\bar{z})}{\psi(z)}\right) + F$$

$$\overline{\psi}(\bar{z}) \cdot \frac{\hat{d}\varphi(\bar{z})}{d\bar{z}} = A\overline{\varphi}(z)\psi(z) + B\varphi(\bar{z})\overline{\psi}(\bar{z}) + F\psi(z)\overline{\psi}(\bar{z})$$

If we make one more transformation, we can get a proof to one more interesting statement. If we add on both sides the expression $\varphi(\bar{z}) \cdot \frac{\hat{d}\overline{\psi}(\bar{z})}{d\bar{z}}$ and given into consideration that

$$\frac{\hat{d}(\overline{\psi}(\bar{z}) \cdot \varphi(\bar{z}))}{d\bar{z}} = \frac{\hat{d}\overline{\psi}(\bar{z})}{d\bar{z}} \cdot \varphi(\bar{z}) + \overline{\psi}(\bar{z}) \cdot \frac{\hat{d}\varphi(\bar{z})}{d\bar{z}}$$

then, we get

$$\frac{\hat{d}(\overline{\psi}(\bar{z}) \cdot \varphi(\bar{z}))}{d\bar{z}} = \frac{\hat{d}\overline{\psi}(\bar{z})}{d\bar{z}} \cdot \varphi(\bar{z}) + A\overline{(\varphi(\bar{z})\overline{\psi}(\bar{z}))} + B \cdot \varphi(\bar{z})\overline{\psi}(\bar{z}) + F\psi(z)\overline{\psi}(\bar{z})$$

So, if

$$\psi(z)\overline{\psi}(\overline{z})=1 \text{ and } \varphi(\overline{z})\cdot\frac{\hat{d}\overline{\psi}(\overline{z})}{d\overline{z}}=0 \qquad (3)$$

we get another solution to the Vekua equation (1), i.e. the function $W_1=\varphi(\overline{z})\cdot\overline{\psi}(\overline{z})$ which is not from the form (2). It is an antyanalytic function. If we want a solution that $W\neq 0$, then $\varphi\left(\overline{z}\right)\neq 0$, which means that the second condition in (3) is $\frac{\hat{d}\overline{\psi}(\overline{z})}{d\overline{z}}=0$.

So, now we can formulate the proven fact as a theorem.

Theorem 1. Let $\varphi=\varphi(\overline{z})$ be an antyanalytic function and $\psi=\psi(z)$ be an analytic function. If $W=\frac{\varphi(\overline{z})}{\psi(z)}$ is a solution to the Vekua equation (1), then $W_1=\varphi(\overline{z})\overline{\psi}(\overline{z})$ is also a solution to the Vekua equation (1), if the conditions (3) are satisfied.

Case 2. Let $W=\frac{\psi(z)}{\varphi(\overline{z})}$ be a solution of the equation (1) (and is from the form (2)), i.e. it is a ratio from one analytic and one antyanalytic function. That means that this function satisfies the equation (1), so if we find the operator derivative by $\overline{z}$ from W and replace it in (1) we get:

$$\frac{\hat{d}W}{d\overline{z}}=\frac{\hat{d}}{d\overline{z}}\left(\frac{\psi(z)}{\varphi(\overline{z})}\right)=\psi(z)\frac{\hat{d}}{d\overline{z}}\left(\frac{1}{\varphi(\overline{z})}\right)=-\frac{\psi(z)}{\varphi^2(\overline{z})}\cdot\frac{\hat{d}\varphi(\overline{z})}{d\overline{z}}$$

$$-\frac{\psi(z)}{\varphi^2(\overline{z})}\cdot\frac{\hat{d}\varphi(\overline{z})}{d\overline{z}}=A\overline{\left(\frac{\psi(z)}{\varphi(\overline{z})}\right)}+B\left(\frac{\psi(z)}{\varphi(\overline{z})}\right)+F$$

$$-\frac{\psi(z)}{\varphi(\overline{z})}\cdot\overline{\varphi}\left(z\right)\cdot\frac{\hat{d}\varphi(\overline{z})}{d\overline{z}}=A\overline{\psi}(\overline{z})\varphi(\overline{z})+B\psi(z)\overline{\varphi}(z)+F\varphi(\overline{z})\cdot\overline{\varphi}\left(z\right)$$

If we make one more transformation, we can get a proof to another interesting statement. Here we expect a new solution of the Vekua equation to be the function $W_1=\overline{\varphi}(z)\cdot\psi(z)$. It is analytic function, so its areolar derivative is 0. So, if we add on both sides the expression $\frac{\hat{d}W_1}{d\overline{z}}$, then we get

$$-\frac{\psi(z)}{\varphi(\overline{z})}\cdot\overline{\varphi}\left(z\right)\cdot\frac{\hat{d}\varphi(\overline{z})}{d\overline{z}}+\frac{\hat{d}(\psi(z)\overline{\varphi}(z))}{d\overline{z}}=A(\overline{\psi(z)\overline{\varphi}(z)})+B\psi(z)\overline{\varphi}(z)+F\varphi(\overline{z})\cdot\overline{\varphi}(z)$$

So, if

$$\varphi(\bar{z})\cdot\bar{\varphi}(z)=1 \text{ and } -\frac{\psi(z)}{\varphi(\bar{z})}\cdot\bar{\varphi}(z)\cdot\frac{\hat{d}\varphi(\bar{z})}{d\bar{z}}=0 \tag{4}$$

we get another solution to the Vekua equation (1), i.e. the function $W_1=\bar{\varphi}(z)\cdot\psi(z)$ which is not from the form (2). It is an analytic function. Again, the second condition in (4), means that $\frac{\hat{d}\varphi(\bar{z})}{d\bar{z}}=0$.

So, now we can formulate the proven fact as a theorem.

Theorem 2. Let $\varphi=\varphi(\bar{z})$ be an antyanalytic function and $\psi=\psi(z)$ be an analytic function. If $W=\frac{\psi(z)}{\varphi(\bar{z})}$ is a solution to the Vekua equation (1), then $W_1=\bar{\varphi}(z)\cdot\psi(z)$ is also a solution to the Vekua equation (1), if the conditions (4) are satisfied.

Case 3. Let $W=\psi(z)\varphi(\bar{z})$ is a solution of the equation (1) (and is from the form (2)), i.e. it is a product from one antyanalytic and one analytic function. That means that this function satisfies the equation (1), so if we find the operator derivative by $\bar{z}$ from W and replace it in (1) we get:

$$\frac{\hat{d}W}{d\bar{z}}=\frac{\hat{d}}{d\bar{z}}(\psi(z)\varphi(\bar{z}))=\psi(z)\cdot\frac{\hat{d}\varphi(\bar{z})}{d\bar{z}}$$

$$\psi(z)\cdot\frac{\hat{d}\varphi(\bar{z})}{d\bar{z}}=A\overline{(\psi(z)\varphi(\bar{z}))}+B(\psi(z)\varphi(\bar{z}))+F$$

$$\frac{1}{\bar{\psi}(\bar{z})}\cdot\frac{\hat{d}\varphi(\bar{z})}{d\bar{z}}=A\frac{\bar{\varphi}(z)}{\psi(z)}+B\frac{\varphi(\bar{z})}{\bar{\psi}(\bar{z})}+F\frac{1}{\psi(z)\bar{\psi}(\bar{z})}$$

If we make one more transformation, we can get a proof to one more interesting statement. If we add on both sides the expression $-\frac{\varphi(\bar{z})}{\bar{\psi}^2(\bar{z})}\cdot\frac{\hat{d}\bar{\psi}(\bar{z})}{d\bar{z}}$ and given into consideration that

$$\frac{\hat{d}}{d\bar{z}}\left(\frac{\varphi(\bar{z})}{\bar{\psi}(\bar{z})}\right)=\frac{1}{\bar{\psi}^2(\bar{z})}\left(\frac{\hat{d}\varphi(\bar{z})}{d\bar{z}}\cdot\bar{\psi}(\bar{z})-\varphi(\bar{z})\frac{\hat{d}\bar{\psi}(\bar{z})}{d\bar{z}}\right)=\frac{1}{\bar{\psi}(\bar{z})}\frac{\hat{d}\varphi(\bar{z})}{d\bar{z}}-\frac{\varphi(\bar{z})}{\bar{\psi}^2(\bar{z})}\cdot\frac{\hat{d}\bar{\psi}(\bar{z})}{d\bar{z}}$$

then, we get

$$\frac{\hat{d}}{d\bar{z}}\left(\frac{\varphi(\bar{z})}{\bar{\psi}(\bar{z})}\right)=A\overline{\left(\frac{\varphi(\bar{z})}{\bar{\psi}(\bar{z})}\right)}+B\cdot\frac{\varphi(\bar{z})}{\bar{\psi}(\bar{z})}+F\frac{1}{\psi(z)\bar{\psi}(\bar{z})}-\frac{\varphi(\bar{z})}{\bar{\psi}^2(\bar{z})}\cdot\frac{\hat{d}\bar{\psi}(\bar{z})}{d\bar{z}}$$

So, if

$$\psi(z)\bar{\psi}(\bar{z})=1 \text{ and } -\frac{\varphi(\bar{z})}{\bar{\psi}^2(\bar{z})}\cdot\frac{\hat{d}\bar{\psi}(\bar{z})}{d\bar{z}}=0 \tag{5}$$

we get another solution to the Vekua equation (1), i.e. the function $W_1 = \frac{\varphi(\bar{z})}{\bar{\psi}(\bar{z})}$ which is not from the form (2). Again, (5) means that $\frac{\hat{d}\bar{\psi}(\bar{z})}{d\bar{z}} = 0$.

So, now we can formulate the proven fact as a theorem.

Theorem 3. Let $\varphi = \varphi(\bar{z})$ be an antyanalytic function and $\psi = \psi(z)$ be an analytic function. If $W = \psi(z)\varphi(\bar{z})$ is a solution to the Vekua equation (1), then $W_1 = \frac{\varphi(\bar{z})}{\bar{\psi}(\bar{z})}$ is also a solution to the Vekua equation (1), if the conditions (5) are satisfied.

Note. As we can see in all three cases, the second twin solution is not from the form (2). The functions are different, but the conditions are similar.

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REMARK ABOUT CHARACTERIZATION OF 2-INNER PRODUCT

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Abstract. Characterization of 2-inner product is focus of interest of many mathematicians. In this paper proofs of two characterizations of 2- inner product, which are actually consequences of the Theorem 1 [15], are given. Also, generalizations of already know Hayashi (see [4], pg. 297) and Zarantonello ([5]) inequalities are fully elaborated.

1. INTRODUCTION

The concepts of 2-norm and 2-inner product are two-dimensional analogies to the concepts of norm and inner product. S. Gähler ([13]), 1965, gave the term of 2-norm and R. Ehret ([11]), 1969, proved the following:

If $(L,(\cdot,\cdot|\cdot))$ is a 2-pre-Hilbert space, then

$$\|x,y\|=(x,x|y)^{1/2}, \qquad (1)$$

for all $x,y\in L$, defines a 2-norm. So, we get the 2-normed space $(L,\|\cdot,\cdot\|)$ and furthermore for all x, y, z ϵ L the following equalities are satisfied:

$$(x,y|z)=\frac{\|x+y,z\|^2-\|x-y,z\|^2}{4}, \qquad (2)$$

$$\|x+y,z\|^2+\|x-y,z\|^2=2(\|x,z\|^2+\|y,z\|^2), \qquad (3)$$

The equality (3) is analogue to the parallelogram equality, and it is said to be parallelepiped equality. Moreover, 2-normed space L is 2-pre-Hilbert if and only if the equality (3) is satisfied for all $x,y,z\in L$.

The papers [1]-[3], [6], [12], [14]-[16] consist of many proven characterizations about 2-inner product.

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Theorem 1 ([15]). Let $(L,\|\cdot,\cdot\|)$ be a real 2-normed space. Then, L is a 2-pre-Hilbert space if and only if the following condition is satisfied:
if $n\geq 3$, $x_1,x_2,...,x_n,z\in L$ and $a_1,a_2,...,a_n$ are real numbers such that $\sum_{i=1}^{n} a_i=0$, then

$$\|\sum_{i=1}^{n} a_i x_i,z\|^2=-\sum_{1\leq i<j\leq n} a_i a_j\|x_i-x_j,z\|^2 . \blacksquare \qquad (4)$$

2. CHARACTERIZATION OF 2-PRE-HILBERT SPACE

The characterization of 2-inneer product by applying the Euler-Lagrange type of equality is elaborated in [6] or in other words generalization of Corollary 2.2 [8], is elaborated. The following theorem is one other proof of the above stated generalization.

Theorem 2 ([6]). Let $(L,\|\cdot,\cdot\|)$ be a real 2-normed space. The 2-norm is generated by 2-inner product if and only if the following equality is satisfied

$$\frac{\|ax+by,z\|^2}{\gamma}+\frac{\|\beta bx-\alpha ay,z\|^2}{\gamma\alpha\beta}=\frac{\|x,z\|^2}{\alpha}+\frac{\|y,z\|^2}{\beta}, \qquad (5)$$

for all $x,y,z\in L$ and for all $a,b\in\mathbf{R}$, $\alpha,\beta>0, \gamma=\alpha a^2+\beta b^2$.

Proof. Let L be a real 2-normed space such that for all $x,y,z\in L$ and for all $a,b\in\mathbf{R}$, $\alpha,\beta>0, \gamma=\alpha a^2+\beta b^2$ the equality (5) is satisfied. For $\alpha=\beta=a=b=1$, the equality (5) is transformed to an equality which is equivalent to the parallelepiped equality, (3), what actually means that L is 2-pre-Hilbert space in which the 2-inner product is defined as (2) and moreover (1) holds true.
Conversely, let 2-inner product, which determines the 2-norm, exist and let $x,y,z\in L$ and $a,b\in\mathbf{R}$, $\alpha,\beta>0$ be such that $\gamma=\alpha a^2+\beta b^2$ is satisfied. For

$$a_1=\frac{a}{\sqrt{\gamma}},\ a_2=\frac{b}{\sqrt{\gamma}},\ a_3=-\frac{a+b}{\sqrt{\gamma}},\ x_1=x,\ x_2=y,\ x_3=0$$

theorem 1 is transformed as the following

$$\frac{\|ax+by,z\|^2}{\gamma}=\frac{a(a+b)}{\gamma}\|x,z\|^2+\frac{b(a+b)}{\gamma}\|y,z\|^2-\frac{ab}{\gamma}\|x-y,z\|^2 . \qquad (6)$$

Further, for

$$a_1 = \frac{b\sqrt{\beta}}{\sqrt{\alpha\gamma}}, a_2 = -\frac{a\sqrt{\alpha}}{\sqrt{\beta\gamma}}, a_3 = -\frac{b\beta-a\alpha}{\sqrt{\alpha\beta\gamma}},\ x_1 = x,\ \ x_2 = y, x_3 = 0$$

theorem 1 is transformed as the following

$$\frac{\|\beta bx-\alpha ay,z\|^2}{\gamma\alpha\beta} = \frac{b(b\beta-a\alpha)}{\alpha\gamma}\| x,z \|^2 - \frac{a(b\beta-a\alpha)}{\beta\gamma}\| y,z \|^2 + \frac{ab}{\gamma}\| x-y,z \|^2 . \tag{7}$$

Finally, if we summarize the equalities (6) and (7) and have also on mind that $\gamma = \alpha a^2 + \beta b^2$ we get the following

$$\begin{aligned}\frac{\|ax+by,z\|^2}{\gamma} + \frac{\|\beta bx-\alpha ay,z\|^2}{\gamma\alpha\beta} &= \left(\frac{a(a+b)}{\gamma} + \frac{b(b\beta-a\alpha)}{\alpha\gamma}\right)\| x,z \|^2 + \\ &\quad + \left(\frac{b(a+b)}{\gamma} - \frac{a(b\beta-a\alpha)}{\beta\gamma}\right)\| y,z \|^2 \\ &= \frac{\alpha a^2+\beta b^2}{\alpha\gamma}\| x,z \|^2 + \frac{\alpha a^2+\beta b^2}{\beta\gamma}\| y,z \|^2 \\ &= \frac{\|x,z\|^2}{\alpha} + \frac{\|y,z\|^2}{\beta},\end{aligned}$$

i.e. the equality (5) is satisfied. ■

The following theorem is actually generalization of M. S. Moslehian and J. M. Rassias (Corollary 2.2, [9]) result.

Theorem 3. A real 2- normed space $(L,\|\cdot,\cdot\|)$ is 2- pre-Hilfert space if and only if for each $n \geq 2$ and for all $x_1, x_2, ..., x_n, z \in L$ the equality (8) is satisfied

$$\sum_{a_i\in\{-1,1\}} \| x_1 + \sum_{i=2}^{n} a_i x_i , z \|^2 = \sum_{a_i\in\{-1,1\}} (\| x_1, z \| + \sum_{i=2}^{n} a_i \| x_i , z \|)^2 . \tag{8}$$

Proof. Let (8) be satisfied for each $n \geq 2$ and for all $x_1, x_2, ..., x_n, z \in L$. For $n = 2$, $x_1 = x$ and $x_2 = y$ the equality (8) is transformed to the parallelepiped equality (3). That actually means that L is 2-pre-Hilbert space in which the 2-inner product is defined as (2) and furthermore (1) holds true.

Conversely, let a 2-normed space $(L,\|\cdot,\cdot\|)$ be a 2-pre-Hilbert space, $n \geq 2$ and $x_1, x_2, ..., x_n, z \in L$.

For $a_{n+1} = -(1 + \sum_{k=2}^{n} a_i)$ and $x_{n+1} = 0$, Theorem 1 is transformed as the following:

$$\begin{aligned}\| x_1+\sum_{i=2}^{n} a_i x_i, z \|^2 &= \| x_1+\sum_{i=2}^{n+1} a_i x_i, z \|^2 \\ &= (1+\sum_{k=2}^{n} a_k)(\| x_1, z\|^2+\sum_{i=2}^{n} a_i \| x_i, z\|^2)- \\ &\quad -\sum_{i=2}^{n} a_i \| x_1-x_i, z\|^2-\sum_{2\le i<j\le n} a_i a_j \| x_i-x_j, z\|^2 \\ &= \sum_{i=1}^{n}\| x_i, z\|^2+\| x_1, z\|^2\sum_{k=2}^{n} a_k+\sum_{k=2}^{n}\sum_{\substack{i=2\\ i\neq k}}^{n} a_k a_i \| x_i, z\|^2- \\ &\quad -\sum_{i=2}^{n} a_i \| x_1-x_i, z\|^2-\sum_{2\le i<j\le n} a_i a_j \| x_i-x_j, z\|^2\end{aligned}$$

and since $a_i \in \{-1,1\}$, for $i=2,3,...,n$, we get 2^{n-1} equalities of the above type. By summarizing the such obtained equalities, we get the following.

$$\begin{aligned}\sum_{a_i\in\{-1,1\}}\| x_1+\sum_{i=2}^{n} a_i x_i, z \|^2 &= 2^{n-1}\sum_{i=1}^{n}\| x_i, z\|^2+\sum_{a_k\in\{-1,1\}}\sum_{k=2}^{n} a_k \| x_1, z\|^2+ \\ &\quad +\sum_{a_k, a_i\in\{-1,1\}}\sum_{k=2}^{n}\sum_{\substack{i=2\\ i\neq k}}^{n} a_k a_i \| x_i, z\|^2- \\ &\quad -\sum_{a_k, a_i\in\{-1,1\}}\sum_{i=2}^{n} a_i \| x_1-x_i, z\|^2- \\ &\quad -\sum_{a_k, a_i\in\{-1,1\}}\sum_{2\le i<j\le n} a_i a_j \| x_i-x_j, z\|^2 \\ &= 2^{n-1}\sum_{i=1}^{n}\| x_i, z\|^2 \\ &= \sum_{a_i\in\{-1,1\}}(\| x_1, z\|+\sum_{i=2}^{n} a_i \| x_i, z\|)^2,\end{aligned}$$

i.e. the equality (8) holds true. ■

3. GENERALIZATION OF HAYASHI AND ZARANTONELLO INEQUALITIES

The following theorems, are actually generalization of two already known equalities, obtained by using theorem 1. Thus, we will firstly give a generalization of Hayashi (see [4], pg. 297) inequality for complex numbers.

Theorem 4. Let $(L,\|\cdot,\cdot\|)$ be a real 2-normed space. Then

$$\sum_{cyclic} \| x-x_1,z\|\cdot\| x-x_2,z\|\cdot\| x_1-x_2,z\| \geq \| x_1-x_2,z\|\cdot\| x_2-x_3,z\|\cdot\| x_3-x_1,z\| \quad (9)$$

for all $x,x_1,x_2,x_3,z\in L$. The inequality is transformed to an equality, if at least one of the sets $\{x-x_1,z\},\{x-x_2,z\},\{x-x_3,z\}$ is linearly dependent or more over if the set

$$\{\tfrac{\|x_2-x_3,z\|}{\|x-x_1,z\|}(x-x_1)+\tfrac{\|x_3-x_1,z\|}{\|x-x_2,z\|}(x-x_2)+\tfrac{\|x_1-x_2,z\|}{\|x-x_3,z\|}(x-x_3),z\}$$

is linearly dependent.

Proof. Let at least one of the sets $\{x-x_1,z\},\{x-x_2,z\},\{x-x_3,z\}$ be linearly dependent. With no loose of the generality, let $\{x-x_1,z\}$ be such the set, i.e. $x=x_1+\alpha z$. Then, the properties of 2-norm imply the following

$$\sum_{cyclic} \| x-x_1,z\|\cdot\| x-x_2,z\|\cdot\| x_1-x_2,z\| = \| x-x_2,z\|\cdot\| x-x_3,z\|\cdot\| x_2-x_3,z\|$$

$$= \| x_1+\alpha z-x_2,z\|\cdot\| x_1+\alpha z-x_3,z\|\cdot\| x_2-x_3,z\|$$

$$= \| x_1-x_2,z\|\cdot\| x_2-x_3,z\|\cdot\| x_3-x_1,z\|,$$

The above means that (9) is an equality.

Let's suppose that the sets $\{x-x_1,z\},\{x-x_2,z\},\{x-x_3,z\}$ are linearly independent. For $a_4=-\sum_{i=1}^{3} a_i$ and $x_4=x$ in Theorem 1, we get that for all $x,x_1,x_2,x_3,z\in L$ and for all $a_1,a_2,a_3\in\mathbf{R}$ the equality

$$\|\sum_{i=1}^{3} a_i x_i - x\sum_{i=1}^{3} a_i, z\|^2 = (\sum_{i=1}^{3} a_i)\cdot(\sum_{i=1}^{3} a_i \| x-x_i,z\|) - \sum_{1\leq i<j\leq 3} a_i a_j \| x_i-x_j,z\|^2$$

holds true.

The right side of the above equality is nonnegative. Therefore, for all $x,x_1,x_2,x_3,z\in L$ and for all $a_1,a_2,a_3\in\mathbf{R}$ the inequality (10) holds true

$$(\sum_{i=1}^{3} a_i)\cdot(\sum_{i=1}^{3} a_i \| x-x_i,z\|) \geq \sum_{1\leq i<j\leq 3} a_i a_j \| x_i-x_j,z\|^2 . \quad (10)$$

For

$$a_1=\tfrac{\|x_2-x_3,z\|}{\|x-x_1,z\|},\ a_2=\tfrac{\|x_3-x_1,z\|}{\|x-x_2,z\|},\ a_3=\tfrac{\|x_1-x_2,z\|}{\|x-x_3,z\|}$$

the inequality (10) is transformed as the followings

$$\sum_{i\neq j\neq k\neq i} \tfrac{\|x_i-x_j,z\|}{\|x-x_k,z\|} \sum_{i\neq j\neq k\neq i} \| x_i-x_j,z\|\cdot\| x-x_k,z\| \geq \sum_{i\neq j\neq k\neq i} \tfrac{\|x_j-x_k,z\|}{\|x-x_i,z\|}\tfrac{\|x_k-x_i,z\|}{\|x-x_j,z\|} \| x_i-x_j,z\|^2$$

$$\sum_{i\neq j\neq k\neq i} \frac{\|x_i - x_j, z\|}{\|x - x_k, z\|} \sum_{i\neq j\neq k\neq i} \| x_i - x_j, z \| \cdot \| x - x_k, z \| \geq$$

$$\geq \frac{\|x_1 - x_2, z\|\cdot\|x_2 - x_3, z\|\cdot\|x_3 - x_1, z\|}{\|x - x_1, z\|\cdot\|x - x_2, z\|\cdot\|x - x_3, z\|} \sum_{i\neq j\neq k\neq i} \| x_i - x_j, z \| \cdot \| x - x_k, z \|$$

$$\sum_{i\neq j\neq k\neq i} \frac{\|x_i - x_j, z\|}{\|x - x_k, z\|} \geq \frac{\|x_1 - x_2, z\|\cdot\|x_2 - x_3, z\|\cdot\|x_3 - x_1, z\|}{\|x - x_1, z\|\cdot\|x - x_2, z\|\cdot\|x - x_3, z\|}.$$

Clearly, the last inequality is equivalent to the inequality (9). The proof implies that the inequality (9) might be transformed to an equality if (10) is an equality, i.e. if the set $\{\sum_{i=1}^{3} a_i x_i - x\sum_{i=1}^{3} a_i, z\}$ is linearly dependent, that is if the set

$$\{\frac{\|x_2 - x_3, z\|}{\|x - x_1, z\|}(x - x_1) + \frac{\|x_3 - x_1, z\|}{\|x - x_2, z\|}(x - x_2) + \frac{\|x_1 - x_2, z\|}{\|x - x_3, z\|}(x - x_3), z\}$$

is linearly dependent. ■

On the end of our considerations we will generalize the Zarantonello ([5]), inequality, i.e. we will prove the following theorem.

Theorem 5. Let L be a real 2-pre-Hilbert space and $f : L \to L$ be a function such that

$$\| f(x) - f(y), z \| \leq \| x - y, z \|, \tag{11}$$

holds true, for all $x, y, x \in L$, Then for all $a_1, a_2, ..., a_n \geq 0$, such that $\sum_{i=1}^{n} a_i = 1$ and for all $y_1, y_2, ..., y_n, z \in L$

$$\| \sum_{i=1}^{n} a_i f(y_i) - f(\sum_{k=1}^{n} a_k y_k), z \|^2 \leq \sum_{1\leq i<k\leq n} a_i a_k (\| y_i - y_k, z \|^2 - \| f(y_i) - f(y_k), z \|^2) \tag{12}$$

holds true.

Proof. For

$$x_i = f(y_i), i = 1,2,..,.n, \; x_{n+1} = f(\sum_{i=1}^{n} a_i y_i)$$

and $a_{n+1} = -1$, in Theorem 1 and then by using the inequality (11) and the properties of 2-norm, we get that for all $a_1, a_2, ..., a_n \geq 0$ such that $\sum_{i=1}^{n} a_i = 1$ and for all $y_1, y_2, ..., y_n,\; z \in L$, the following holds true

$$\|\sum_{i=1}^{n} a_i f(y_i) - f(\sum_{k=1}^{n} a_k y_k), z\|^2 =$$

$$= \sum_{i=1}^{n} a_i \|f(y_i) - f(\sum_{k=1}^{n} a_k y_k), z\|^2 - \sum_{1\le i<k\le n} a_i a_k \| f(y_i) - f(y_k), z\|^2$$

$$\le \sum_{i=1}^{n} a_i \|y_i - \sum_{k=1}^{n} a_k y_k, z\|^2 - \sum_{1\le i<k\le n} a_i a_k \| f(y_i) - f(y_k), z\|^2$$

$$= \sum_{i=1}^{n} a_i \|\sum_{k=1}^{n} a_k y_i - \sum_{k=1}^{n} a_k y_k, z\|^2 - \sum_{1\le i<k\le n} a_i a_k \| f(y_i) - f(y_k), z\|^2$$

$$= \sum_{i=1}^{n} a_i \|\sum_{k=1}^{n} a_k (y_i - y_k), z\|^2 - \sum_{1\le i<k\le n} a_i a_k \| f(y_i) - f(y_k), z\|^2.$$

On the other hand, for $x_k = y_i - y_k, k = 1,2,...,n, x_{n+1} = 0$ and $a_{n+1} = -1$ in Theorem 1 and also by using that $a_i \ge 0$, for $i = 1,2,...,n$, we get that

$$\|\sum_{k=1}^{n} a_k (y_i - y_k), z\|^2 = \sum_{k=1}^{n} a_k \| y_i - y_k, z\|^2 - \sum_{1\le i<j\le n} a_i a_j \| y_i - y_j, z\|^2$$

$$\le \sum_{k=1}^{n} a_k \| y_i - y_k, z\|^2 .$$

Finally, the last two inequalities imply the inequality (12). ■

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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ABOUT THE ACCORDANCE BETWEEN THE CANONICAL VEKUA DIFFERENTIAL EQUATION AND THE GENERALIZED HOMOGENEOUS DIFFERENTIAL EQUATION

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Abstract. In the paper two equations, the canonical Vekua differential equation and the generalized homogeneous differential equation, are considered. The main result is the theorem with the condition for the accordance between this two equations.

1. INTRODUCTION

The equation

$$\frac{\hat{d}W}{d\bar{z}} = AW + B\overline{W} + F \tag{1}$$

where $A = A(z)$, $B = B(z)$ and $F = F(z)$ are given complex functions from a complex variable $z \in D \subseteq \mathbb{C}$ is the well known Vekua equation [1] according to the unknown function $W = W(z) = u + iv$. The derivative on the left side of this equation has been introduced by G.V. Kolosov in 1909 [2]. During his work on a problem from the theory of elasticity, he introduced the expressions

$$\frac{1}{2}[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})] = \frac{\hat{d}W}{dz} \tag{2}$$

and

$$\frac{1}{2}[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})] = \frac{\hat{d}W}{d\bar{z}} \tag{3}$$

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known as operator derivatives of a complex function $W = W(z) = u(x, y) + iv(x, y)$ from a complex variable $z = x + iy$ and $\overline{z} = x - iy$ corresponding. The operating rules for this derivatives are completely given in the monograph of Г. Н.Положий [3] (page18-31). In the mentioned monograph are defined so cold operator integrals $\hat{\int} f(z)dz$ and $\hat{\int} f(z)d\overline{z}$ from $z = x + iy$ and $\overline{z} = x - iy$ corresponding (page 32-41). As for the complex integration in the same monograph is emphasized that it is assumed that all operator integrals can be solved in the area D.

In the Vekua equation (1) the unknown function $W = W(z)$ is under the sign of a complex conjugation which is equivalent to the fact that $B = B(z)$ is not identically equaled to zero in D. That is why for (1) the quadratures that we have for the equations where the unknown function $W = W(z)$ is not under the sign of a complex conjugation, stop existing.

This equation is important not only for the fact that it came from a practical problem, but also because depending on the coefficients A, B and F the equation (1) defines different classes of generalized analytic functions. For example, for $F = F(z) \equiv 0$ in D the equation (1) i.e.

$$\frac{\hat{d}W}{d\overline{z}} = AW + B\overline{W} \tag{4}$$

which is called canonical Vekua equation, defines so cold generalized analytic functions from fourth class; and for $A \equiv 0$ and $F \equiv 0$ in D, the equation (1) i.e. the equation $\frac{\hat{d}W}{d\overline{z}} = B\overline{W}$ defines so cold generalized analytic functions from third class or the (r+is)-analytic functions [3], [4].

Those are the cases when $B \neq 0$. But if we put $B \equiv 0$, we get the following special cases. In the case $A \equiv 0$, $B \equiv 0$ and $F \equiv 0$ in the working area $D \subseteq \mathbb{C}$ the equation (1) takes the following expression $\frac{\hat{d}W}{d\overline{z}} = 0$ and this equation, in the class of the functions $W = u(x, y) + iv(x, y)$ whose real and imaginary parts have unbroken partial derivatives u'_x, u'_y, v'_x and v'_y in D, is a complex writing of the Cauchy - Riemann conditions. In other words it defines the analytic functions in the sense of the classic theory of the analytic functions. In the case $B \equiv 0$ in D i.e. $\frac{\hat{d}W}{d\overline{z}} = AW + F$ is the so cold areolar linear differential equation [3] (page 39-40) and it can be solved with quadratures by the formula:

$$W = e^{\hat{\int} A(z)d\bar{z}}[\Phi(z) + \hat{\int} F(z) e^{-\hat{\int} A(z)d\bar{z}} d\bar{z}].$$

Here $\Phi = \Phi(z)$ is an arbitrary analytic function in the role of an integral constant.

2. FORMULATION OF THE PROBLEM AND MAIN RESULT

In the paper [5], the following lemma is proved.

Lemma. The equations

$$\frac{\hat{d}W}{dz} = f(z,W) \tag{5}$$

and

$$\frac{\hat{d}W}{d\bar{z}} = g(z,W) \tag{6}$$

where $\frac{\hat{d}f}{d\overline{W}} = 0$, have common solutions if and only if

$$\frac{\hat{d}f}{d\bar{z}} + \frac{\hat{d}f}{dW} g = \frac{\hat{d}g}{dz} + \frac{\hat{d}g}{dW} f + \frac{\hat{d}g}{d\overline{W}} \bar{g} \;. \tag{7}$$

It is assumed that the operator derivatives in (7) exist and that they are continuous functions in the working area D from the complex plane.

In this paper we are examining the accordance between the canonical Vekua equation (4), on one side and the generalized homogeneous differential equation

$$\frac{\hat{d}W}{dz} = \varphi(\frac{W}{z}) \tag{8}$$

on the other side, where $\varphi = \varphi(z)$ is a given complex functions from a complex variable $z \in D \subseteq \mathbb{C}$, such that $\frac{\hat{d}\varphi}{d\overline{W}} = 0$. The canonical Vekua equation (4) is an equation of type (6) where

$$g(z,W) = AW + B\overline{W} \tag{9}$$

and the generalized homogeneous differential equation (8) is an equation of type (5), where

$$f(z,W) = \varphi(\frac{W}{z}) \;. \tag{10}$$

Here, the function f is an analytic function according to W, which means that $\frac{\hat{d}f}{d\overline{W}}=0$. That is the only condition to be accomplished, so that we can use the mentioned lemma.

If we calculate all the derivatives in (7), we get that

$$\frac{\hat{d}f}{d\overline{z}}=\frac{\hat{d}\varphi}{d\overline{z}}, \qquad \frac{\hat{d}f}{dW}=\frac{\hat{d}}{dW}\varphi(\tfrac{W}{z})=\frac{\hat{d}\varphi}{dW}\cdot\frac{1}{z},$$

$$\frac{\hat{d}g}{dz}=\frac{\hat{d}A}{dz}W+\frac{\hat{d}B}{dz}\overline{W}, \qquad \frac{\hat{d}g}{dW}=A, \qquad \frac{\hat{d}g}{d\overline{W}}=B.$$

And if we put them in (7) we get that

$$\frac{\hat{d}\varphi}{d\overline{z}}+\frac{\hat{d}\varphi}{dW}\cdot\frac{1}{z}(AW+B\overline{W})=\frac{\hat{d}A}{dz}W+\frac{\hat{d}B}{dz}\overline{W}+A\varphi+B\overline{(AW+B\overline{W})}.$$

Now we write the last equation in the following form

$$W(\frac{\hat{d}\varphi}{dW}\cdot\frac{A}{z}-\frac{\hat{d}A}{dz}-|B|^2)+\overline{W}(\frac{\hat{d}\varphi}{dW}\cdot\frac{B}{z}-\frac{\hat{d}B}{dz}-\overline{A}B)+\frac{\hat{d}\varphi}{d\overline{z}}-A\varphi=0.$$

This linear combination is true only if the following system of equation is satisfied

$$\begin{cases}\frac{\hat{d}\varphi}{dW}\cdot\frac{A}{z}-\frac{\hat{d}A}{dz}-|B|^2=0\\ \frac{\hat{d}\varphi}{dW}\cdot\frac{B}{z}-\frac{\hat{d}B}{dz}-\overline{A}B=0\\ \frac{\hat{d}\varphi}{d\overline{z}}-A\varphi=0\end{cases}. \qquad (11)$$

If we eliminate the derivative $\frac{\hat{d}\varphi}{dW}$ from the first and the second equation in the system (11), we get

$$\frac{\hat{d}\varphi}{dW}=\frac{z}{A}(\frac{\hat{d}A}{dz}+|B|^2)$$

$$\frac{z}{A}(\frac{\hat{d}A}{dz}+|B|^2)\cdot\frac{B}{z}-\frac{\hat{d}B}{dz}-\overline{A}B=0$$

$$B\frac{\hat{d}A}{dz}-A\frac{\hat{d}B}{dz}+B|B|^2-B|A|^2=0$$

Because of the fact that $\frac{\hat{d}}{dz}(\frac{A}{B})=\frac{1}{B^2}(\frac{\hat{d}A}{dz}B-A\frac{\hat{d}B}{dz})$, we get that

$$B^2\frac{\hat{d}}{dz}(\frac{A}{B})+B(|B|^2-|A|^2)=0$$

$$\frac{\hat{d}}{dz}(\frac{A}{B})+\frac{|B|^2-|A|^2}{B}=0$$

or

$$\frac{\hat{d}}{dz}(\frac{A}{B})+\overline{B}-\frac{A}{B}\overline{A}=0 \qquad (12)$$

which is the condition between the coefficients in the Vekua equation (4) in order to has common solutions with the equation (8).

The third equation in the system (11) is an areolar equation which can be solved, i.e.

$$\frac{1}{\varphi}\cdot\frac{\hat{d}\varphi}{d\overline{z}} = A$$

$$\ln\varphi\frac{W}{z} = \hat{\int} A(z)d\overline{z} + \ln\Phi\left(z\right)$$

$$\varphi(\frac{W}{z}) = \Phi\left(z\right)\cdot\exp(\hat{\int} A(z)d\overline{z}) \qquad (13)$$

Here $\Phi = \Phi(z)$ is an arbitrary analytic function in the role of an integral constant.

So, we have proved the following

Theorem. The Vekua equation (4) and the generalized homogeneous equation (8) have common solutions if and only if the condition (12) is fulfilled and the relation between the coefficients of the two equations are given with (13).

Note 1. The condition (12) that we got, works for example if $A = B$.

Note 2. In [6], we can see the condition between the coefficients in the Vekua equation (1) (and (4) also), in order to has common solutions with the generalized linear equation and the relation between its coefficients. If we compare the theorems, they have similar statement, but different conditions and relations that we mentioned. Further more, in [6] both the equations (1) and (4) are considered and in this case only the equation (4) is considered. This refers to the easier manipulation with the generalized linear equation in comparison with the generalized homogeneous equation.

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ON (3,2,ρ)-*S*-*K*-METRIZABLE SPACES

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Abstract. For a given $(3,2,\rho)$- metric d on a set M, we show that any $(3,2,\rho)$-*S*-*K*-metrizable space has an open refinement which is both locally finite and σ-discrete

1. INTRODUCTION

If we review historically the geometric properties, their axiomatic classification and the generalization of metric spaces we can see that, they have been subject of interest of great number of mathematicians and from their work a lot of have been developed. We will mention some of them: K. Menger ([14]), V. Nemytzki, P. S. Aleksandrov ([16], [1]), Z. Mamuzic ([13]), S. Gähler ([11]), A. V. Arhangelskii, M. Choban, S. Nedev ([2], [3], [17]), R. Kopperman ([12]), J. Usan ([18]), B. C. Dhage, Z. Mustafa, B. Sims ([6], [15]). The notion of (n,m,ρ)-metric is introduced in [7]. Connections between some of the topologies induced by a $(3,1,\rho)$-metric d and topologies induced by a pseudo-o-metric, o-metric and symmetric are given in [8]. For a given $(3,j,\rho)$-metric d on a set M, $j\in\{1,2\}$, seven topologies $\tau(G,d),\tau(H,d),\tau(D,d),\tau(N,d),\tau(W,d),\tau(S,d)$ and $\tau(K,d)$ on M, induced by d, are defined in [4], and several properties of these topologies are shown.

In this paper we consider only the topologies $\tau(S,d)$ and $\tau(K,d)$ induced by a $(3,2,\rho)$-metric d and for $\tau=\tau(S,d)=\tau(K,d)$ we prove that any open cover of a $(3,2,\rho)$-*S*-*K*-metrizable space (M,τ) has: a) an open refinement which is both locally finite and σ-discrete, b) σ-discrete base, and c) a $(3,2,\rho)$-*S*-K-metri-

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zable space (M,τ) is perfectly normal.

2. SOME PROPERTIES OF $(3,2,\rho)$-S-K-METRIZABLE SPACES

In this part we state the notions (defined in [4]) used later.

Let M be a nonempty set, and let $d: M^3 \to \mathbb{R}_0^+ = [0,\infty)$. We state four conditions for such a map.

$(\boldsymbol{M}0)$ $d(x,x,x)=0$, for any $x \in M$;

$(\boldsymbol{P})$ $d(x,y,z)=d(x,z,y)=d(y,x,z)$, for any $x,y,z \in M$;

$(\boldsymbol{M}1)$ $d(x,y,z) \le d(x,y,a)+d(x,a,z)+d(a,y,z)$, for any $x,y,z,a \in M$; and

$(\boldsymbol{M}2)$ $d(x,y,z) \le d(x,a,b)+d(a,y,b)+d(a,b,z)$, for any $x,y,z,a,b \in M$.

For a map d as above let $\rho = \{(x,y,z) \mid (x,y,z) \in M^3, d(x,y,z)=0\}$. The set ρ is a $(3,j)$-equivalence on M, as defined and discussed in [7], [4]. The set $\Delta = \{(x,x,x) \mid x \in M\}$ is a $(3,j)$-equivalence on M, $j=1,2$, and the set $\nabla = \{(x,x,y) \mid x,y \in M\}$ is a $(3,1)$-equivalence, but it is not a $(3,2)$-equivalence on M. The condition $(\boldsymbol{M}0)$ implies that $\Delta \subseteq \rho$.

Definition 1. Let $d: M^3 \to \mathbb{R}_0^+ = [0,\infty)$ and ρ be as above. If d satisfies $(\boldsymbol{M}0)$, $(\boldsymbol{P})$ and $(\boldsymbol{M}2)$, we say that d is a $(3,2,\rho)$-metric on M.

Let d be a $(3,2,\rho)$-metric on M, $x \in M$ and $\varepsilon > 0$. As in [4], we consider the following $\varepsilon-$ball, as subsets of M:

$L(x,\varepsilon) = \{y | y \in M, d(x,y,y) < \varepsilon\}$ -“little” ε-ball with center in x and radius ε.

Among the others, a $(3,2,\rho)$-metric d on M induces the following topologies as in [4]:

1) $\tau(K,d)$-the topology generated by all the ε-balls $L(x,\varepsilon)$, i.e. the topology whose base is the set of the finite intersections of ε-balls $L(x,\varepsilon)$;
2) $\tau(S,d)$-the topology defined by: $U \in \tau(S,d)$ iff $\forall x U$, $\exists \varepsilon > 0$ such that $L(x,\varepsilon) \subseteq U$.

Proposition 1. The ball $L(x,\varepsilon) \in \tau(S,d)$, for any x on M and $\varepsilon > 0$.

Proof. It is enough to show that for any $y \in L(x,\varepsilon)$ there is $\delta > 0$, such that $L(y,\delta) \subseteq L(x,\varepsilon)$. Let $y \in L(x,\varepsilon)$ and $\delta = (\varepsilon - d(x,y,y))/4$. Then, for any $z \in L(y,\delta)$ we have:

$$\begin{aligned} d(x,z,z) &\le d(x,y,y) + 2d(z,y,y) \\ &\le d(x,y,y) + 4d(y,z,z) \\ &< d(x,y,y) + 4\delta = \varepsilon. \end{aligned}$$

This implies that $z \in L(x,\varepsilon)$, i.e. $z \in L(y,\delta) \subseteq L(x,\varepsilon)$.

From the proposition 1, it follows that $\tau(S,d) = \tau(K,d)$, for any $(3,2,\rho)$-metric d on M.

Definition 2. We say that a topological space (M,τ) is $(3,2,\rho)$-*S*-*K*-*metrizable* via a $(3,2,\rho)$-metric d on M, if $\tau = \tau(S,d) = \tau(K,d)$.

In the following theorem, one of the most important properties of $(3,2,\rho)$-*S*-*K*-metrizable space is established.

Let (M,τ) be a $(3,2,\rho)$-*S*-*K*-metrizable topological space.

Proposition 2. *Any open cover of a* $(3,2,\rho)$*-S-K-metrizable space has an open refinement which is both locally finite and* σ*-discrete.*

Proof. Let d be a $(3,2,\rho)$-metric on M and $\tau = \tau(S,d) = \tau(K,d)$. Let $\mathcal{U} = \{U_s \mid s \in S\}$ be an open cover of a $(3,2,\rho)$-*S*-*K*-metrizable space (M,τ), and let < be a well-ordering relation on the set S. Define inductively families $\mathcal{V}_n = \{V_{s,n} | s \in S, n \in \mathbb{N}\}$ of subsets of (M,τ) by letting

$$V_{s,n} = \cup L(x, 1/10^n),$$

where the union is taken over all points $x \in M$ satisfying the following conditions:

$$s \text{ is the smallest element of } S \text{ such that } x \in U_s, \tag{1}$$

$$x \notin V_{t,j} \text{ for } j < n \text{ and } t \in S, \tag{2}$$

$$L(x, 11/10^n) \subseteq U_s. \tag{3}$$

It follows from the definition of $V_{s,n}$ that the sets $V_{s,n}$ are open, and (3) implies that $V_{s,n} \subseteq U_s$. Let $y \in M$. Let s be the smallest element of S such that

$y \in U_s$. Then there is $n \in \mathbb{N}$ such that $L(y, 11/10^n) \subseteq U_s$. It is clear that, we have $y \in V_{t,j}$ for $j < n$ and a $t \in S$ or $y \in V_{s,n}$. Hence, the union $\mathcal{V} = \cup\{\mathcal{V}_n \mid n \in \mathbb{N}\}$ is an open refinement of the cover $\mathcal{U} = \{U_s \mid s \in S\}$.

We will prove that for any $n \in \mathbb{N}$ if $y_1 \in V_{s_1,n}$, $y_2 \in V_{s_2,n}$, and $s_1 \neq s_2$, then

$$d(y_1, y_2, y_2) > 1/10^n \text{ and } d(y_1, y_1, y_2) > 1/10^n, \tag{4}$$

and this will show that the families $\mathcal{V}_n$ are discrete, because any $1/10^{n+1}$-L-ball meets at most one member of $\mathcal{V}_n$.

Let $s_1 < s_2$. By the definition of $V_{s_1,n}$ and $V_{s_2,n}$ there are points x_1 and x_2 satisfying (1), (2) and (3) given above, such that $y_1 \in L(x_1, 1/10^n) \subseteq V_{s_1,n}$ and $y_2 \in L(x_2, 1/10^n) \subseteq V_{s_2,n}$. From (3) it follows that $L(x_1, 11/10^n) \subseteq U_{s_1}$, and from (1) we see that $x_2 \notin U_{s_1}$. Hence, $d(x_1, x_2, x_2) \geq 11/10^n$. The inequalities

$$\begin{aligned} 11/10^n &\leq d(x_1, x_2, x_2) \leq d(x_1, y_1, y_1) + 2d(x_2, y_1, y_1) \\ &\leq d(x_1, y_1, y_1) + 2d(x_2, y_2, y_2) + 4d(y_1, y_2, y_2) \\ &< 3/10^n + 4d(y_1, y_2, y_2), \end{aligned}$$

imply that $d(y_1, y_2, y_2) > 2/10^n > 1/10^n$.

Also the inequalities,

$$\begin{aligned} 11/10^n &\leq d(x_1, x_2, x_2) \leq d(x_1, y_1, y_1) + 2d(x_2, y_1, y_1) \\ &\leq d(x_1, y_1, y_1) + 2d(x_2, y_2, y_2) + 4d(y_1, y_2, y_2) \\ &< 3/10^n + 4d(y_1, y_2, y_2) \leq 3/10^n + 8d(y_1, y_1, y_2), \end{aligned}$$

imply that $d(y_1, y_1, y_2) > 1/10^n$.

From the latter follows the proof of (4).

Furthermore, it is enough to show that for each $t \in S$ and for each pair $k, j \in \mathbb{N}$,

$$\text{if } L(y, 1/10^k) \subseteq V_{t,j} \text{ then } L(y, 1/10^{k+j}) \cap V_{s,n} = \emptyset$$
$$\text{for } n \geq k + j \text{ and } s \in S. \tag{5}$$

From the definition of $V_{s,n}$ we have $V_{s,n} \subseteq V_{s,n+1}$, then $V_{t,j} \subseteq V_{t,j+1}$ for each $t \in S$ and each $j \in \mathbb{N}$. From (2) it follows that each x of $V_{s,n} = \cup L(x, 1/10^n)$, $x \notin V_{t,j}$ for $j < n$. Hence, for $n \geq k + j > k$ and $L(y, 1/10^k) \subseteq$

$V_{t,j}$ it follows that for each x of the union $\cup L(x,1/10^n)$, $d(y,x,x) > 1/10^k$. We will show that $L(y,1/10^{k+j}) \cap V_{s,n} \neq \varnothing$ for $n < k+j$. Let $y \in M$, then there are k, j, t such that $L(y,1/10^k) \subseteq V_{t,j}$ and $n < k+j$. For $\mathcal{V}_m$ discrete family and $m < k+j$, there is δ_m such that $L(y,\delta_m) \cap V_{s,m} \neq \varnothing$ for one s,m. Let $\delta = min\{\delta_m | 1 \leq m < k+j\}$ then $L(y,\delta) \cap V_{s,m} \neq \varnothing$ for all $m < k+j$, i.e. $L(y,1/10^{k+j}) \cap V_{s,n} \neq \varnothing$ for $n < k+j$. From the latter it follows that $\mathcal{V}_n$ is σ-discrete and locally finite. Hence, $\mathcal{V} = \cup\{\mathcal{V}_n | n \in \mathbb{N}\}$ is σ-discrete and locally finite.

Proposition 3. Any $(3,2,\rho)$-*S*-*K*-metrizable space (M,τ) has a σ-discrete base.

Proof. Let d be a $(3,2,\rho)$-metric on M and $\tau = \tau(S,d) = \tau(K,d)$. For $n \in \mathbb{N}$, let $\mathcal{U}_n = \{L(x,1/n) | xM\}$ be an open cover of M and let $\mathcal{V}_n$ be σ-discrete refinement obtained in proposition 2. The definition of $\tau(S,d)$ and $\tau(K,d)$ implies that $\mathcal{U} = \cup\{\mathcal{U}_n | n \in \mathbb{N}\}$ is a base for $\tau = \tau(S,d) = \tau(K,d)$. Since each $\mathcal{V}_n$ is σ-discrete refinement of $\mathcal{U}_n$, it follows that $\mathcal{V} = \cup\{\mathcal{V}_n | n \in \mathbb{N}\}$ is σ-discrete refinement of $\mathcal{U}$. Hence, $\mathcal{V}$ is σ-discrete base of (M,τ).

Corollary 1. Any $(3,2,\rho)$-*S*-*K*-metrizable space (M,τ) has a σ-locally finite base.

We will prove that the existence of a σ-locally finite base is also sufficient for metrizability of a $(3,2)$-*S*-*K-metrizable space* (M,τ).

Proposition 4. Any $(3,2)$-*S*-*K*-metrizable space (M,τ) is perfectly normal.

Proof. Let d be a $(3,2)$-metric on M and $\tau = \tau(S,d) = \tau(K,d)$. Let $\mathcal{V} = \cup\{\mathcal{V}_n | n \in \mathbb{N}\}$, where the families $\mathcal{V}_n$ are a locally finite, be a base for a space (M,τ). Consider an arbitrary open set $W \subseteq M$. For any $x \in W$ there is a natural number $n(x)$ and an open set $U_x \in \mathcal{V}_{n(x)}$ such that $x \in U_x \subseteq \overline{U_x} \subseteq W$. Letting $W_n = \cup\{U_x | n(x) = n\}$ we obtain a sequence $W_1, W_2, \cdots W_n$, $n \in \mathbb{N}$ of open subsets of M such that $W = \cup\{W_n | n \in \mathbb{N}\}$ and by property: if $\{A_s\}_{s \in S}$ is a locally finite family, then the family $\{\overline{A_s}\}_{s \in S}$ also is locally finite, we have

$\overline{W_n} \subseteq W$ for $n \in \mathbb{N}$. Normality of (M,τ) is proven in [9]. Since $W = \cup\{\overline{W_n} \mid n \in \mathbb{N}\}$, the space (M,τ) is perfectly normal.

Proposition 5. If $\rho = \Delta$, then (M,τ) is metrizable.

Proof. For $\rho = \Delta$, and the fact that (M,τ) is regular and has σ-locally finite base then from the metrization theorem of Nagata-Smirnov it follows that (M,τ) is metrizable.

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ALGEBRAIC REPRESENTATION OF A CLASS OF HOMOGENOUS STEINER QUADRUPLE SYSTEMS

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Abstract A Steiner system $S(t,k,v)$ is a pair (Q,B) of v-element set Q and a collection B of its k-element subsets (blocks), such that every t-element subset of Q is contained in exactly one block. Systems $S(2,3,v)$ are Steiner triple systems (STS) and their algebraic representatives are the idempotent totally symmetric quasigroups. Steiner quadruple systems (SQS) are systems $S(3,4,v)$, represented by the idempotent totally symmetric ternary quasigroups.

For $SQS\ (Q,B)$ and $a \in Q$, by taking the set $Q \setminus \{a\}$ and the blocks $\{\{x,y,z\} | \{x,y,z,a\} \in B\}$, a derived triple system is obtained. An SQS is called homogenous if all of its derived triple systems are isomorphic.

In this paper sufficient conditions for SQS to be homogenous are given, resulting with an algebraic representation of one class of homogenous quadruple systems.

1. INTRODUCTION

A Steiner system $S(t,k,v)$ is a pair (Q,B), where Q is a v-element set and B is a collection of its k-element subsets (called blocks) with the property that every t-element subset of Q is contained in a unique block of B. Systems $S(2,3,v)$ and $S(3,4,v)$ are called Steiner triple system (STS) and Steiner quadruple system (SQS) respectively.

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There is a natural connection between $SQS(v)$ and $STS(v-1)$. Let (Q,B) be an $SQS(v)$ and choose an arbitrary element $a \in Q$. Let B_a be the collection of the 3-element subsets of $Q \setminus \{a\}$ which is obtained by selecting all of the blocks of B containing the element a, and then excluding this element from them. Then the pair $(Q \setminus \{a\}, B_a)$ is an $STS(v-1)$. Such a Steiner triple system is called a derived triple system (DTS) of the quadruple system (Q,B). The problem whether or not every STS is a DTS of some quadruple system is open.

Woolhouse in 1844 [12] posed the question: for which integers t, k, and v, does an $S(t,k,v)$ exist? Up to the present time, this problem is also unsolved in general. However, several partial answers are given. Three years later, Kirkman [7] showed that $STS(v)$ exists if and only if $v \equiv 1 \text{ or } 3 \pmod 6$, and constructed systems $S(3,4,2^n)$, for every n. During the late 19^{th} and early 20^{th} century very much was written on the subject of STS, and very little on SQS. Hanani [4] proved that the necessary condition $v \equiv 2 \text{ or } 4 \pmod 6$ for the existence of an SQS of order v is also sufficient, by induction and using six recursive constructions. However, the most extensive study of Steiner systems probably was done in the 70s and the 80s of the last century, concerning the various constructions of a single system and classes of certain type, isomorphism problems, groups of automorphisms, classifications and enumerations, embedings and partial systems, as well as their applications. The development of the computers played significant role, especially in the past thirty years.

Steiner triple and quadruple systems possess "algebraic twins". Given an $STS(v)$ (Q,B), one can define a binary operation $*$ on Q by $a*b=c$ whenever $\{a,b,c\} \in B$, and $a*a=a$. As there is a unique triple in B containing two distinct elements, this operation is well defined. The groupoid $(Q,*)$ belongs to the variety determined by the identities:

$$x \cdot x = x$$
$$x \cdot y = y \cdot x$$
$$x \cdot (x \cdot y) = y.$$

Its members are idempotent totally symmetric quasigroups, also known as Steiner quasigroups, since there is a two-way relationship between such quasigroups and Steiner triple systems. Namely, if $(Q,\cdot)$ is a v-element Steiner quasigroup, then the sets $\{a,b,a\cdot b\}$, $a,b\in Q$, $a\neq b$, are precisely the blocks of an $STS(v)$.

A similar correspondence exists between Steiner quadruple systems and idempotent totally symmetric ternary quasigroups (Steiner 3-quasigroups). They are defined by the following identities:

$$f(x,x,y)=y$$
$$f(x,y,z)=f(x,z,y)=f(y,x,z)$$
$$f(x,y,f(x,y,z))=z.$$

If (Q,B) is an SQS, then the ternary operation f on Q defined by the rules $f(a,a,b)=f(a,b,a)=f(b,a,a)=b$ and $f(a,b,c)=d$ if and only if $\{a,b,c,d\}\in B$ is well defined and satisfies the above identities. On the other hand, if (Q,f) is a finite Steiner 3-quasigroup, then (Q,B) is an SQS, for B consisting of the sets $\{a,b,c,f(a,b,c)\}$, where a,b and c are distinct elements of Q (see [1]). For the quadruple system (Q,B) corresponding to the Steiner 3-quasigroup (Q,f), we will say that it is induced by (Q,f). Note that for $a\in Q$, the DTS $(Q\setminus\{a\},B_a)$ of (Q,B) has an algebraic equivalent $(Q\setminus\{a\},\cdot)$ whose operation can be defined by the ternary operation f, according to the rules

$$x\cdot y=\begin{cases} f(x,y,a), & x\neq y\\ x, & x=y.\end{cases}$$

We also say that the triple system $(Q\setminus\{a\},B_a)$ is induced by $(Q\setminus\{a\},\cdot)$.

An isomorphism from a Steiner system (Q_1,B_1) with parameters t,k,v onto a Steiner system (Q_2,B_2) of the same type is a bijection $\varphi:Q_1\to Q_2$ which maps the k-tuples of Q_1 onto k-tuples of Q_2. An automorphism of (Q,B) is an isomorphism of (Q,B) onto itself. If φ is an isomorphism from a Steiner quasigroup $(Q_1,\cdot)$ onto a Steiner quasigroup $(Q_2,*)$, then φ is also an isomorphism from the corresponding STS (Q_1,B_1) of $(Q_1,\cdot)$ onto the corresponding STS (Q_2,B_2) of $(Q_2,*)$. Namely, if $\{a,b,c\}\in B_1$ then $\varphi(c)=\varphi(a\cdot b)=\varphi(a)*\varphi(b)$, meaning that $\{\varphi(a),\varphi(b),\varphi(c)\}\in B_2$. The same property also holds for SQS. We use it to obtain an algebraic characterization of one class of homogenous Steiner quadruple systems.

The stated relations between Steiner systems and Steiner quasigroups are of great importance for both combinatorial and algebraic structures. Some properties are easier to be proved algebraicly, and others combinatorially. If we prove one in either way, then the corresponding property can be applied to the other structure. In our paper we use algebraic tools to obtain the desired combinatorial property.

2. DESCRIPTION OF A CLASS OF HOMOGENOUS SQS

Given an SQS, let β denote the number of pairwise non-isomorphic derived triple systems. Obviously, $1 \le \beta \le v$, for any $SQS(v)$. The least v for which $\beta > 1$ is 14. There are 4 non-isomorphic $SQS(14)$, and for two of them $\beta = 1$, while for the other two $\beta = 2$ (see [8]). It is clear that these are the only two possible values for β since there are exactly two non-isomorphic $STS(13)$.

Although infinite classes of SQS with $\beta \ge 2$ were constructed, as well as $SQS(v)$ with $\beta \ge t$ for any positive integer t (v much greater than t), the question of determination of β for any given SQS is very far from solved. Moreover, no one as yet has found an order v such that for every k, $1 \le k \le v$, there is an $SQS(v)$ having $\beta = k$.

An SQS is said to be homogenous if its value of β is one, or equivalently if all of its derived triple systems are isomorphic. If all the DTS of an $SQS(v)$ are pairwise non-isomorphic ($\beta = v$), then the quadruple system is called heterogenous.

In what follows, we give an algebraic description of one class of homogenous Steiner quadruple systems.

Lemma. *Let (Q, f) be a finite Steiner 3-quasigroup and φ be an automorphism of Q. Then for $a \in Q$, the derived triple systems $(Q \setminus \{a\}, B_a)$ and $(Q \setminus \{\varphi(a)\}, B_{\varphi(a)})$ of the quadruple system induced by (Q, f) are isomorphic.*

Proof. Let (Q,B) be the *SQS* induced by (Q,f) and define $\psi : Q\setminus\{a\} \to Q\setminus\{\varphi(a)\}$ by $\psi(u)=\varphi(u)$. Let $(Q\setminus\{a\},\circ)$ and $(Q\setminus\{\varphi(a)\},*)$ be the Steiner quasigroups which induce the triple systems $(Q\setminus\{a\},B_a)$ and $(Q\setminus\{\varphi(a)\},B_{\varphi(a)})$ respectively. It is clear that ψ is a bijection, since φ is a bijection. If $u,v\in Q\setminus\{a\}$, and $u\neq v$, then

$$\begin{aligned}\psi(u\circ v)&=\psi(f(u,v,a))=\varphi(f(u,v,a))\\&=f(\varphi(u),\varphi(v),\varphi(a))\\&=\varphi(u)*\varphi(v)=\psi(u)*\psi(v).\end{aligned}$$

The equality $\psi(u\circ u)=\psi(u)=\psi(u)*\psi(u)$ completes the proof.

Corrolary 1. *Le t (Q,f) be a finite Steiner 3-quasigroup with the property that for every $a,b\in Q$, there is an automorphism φ such that $\varphi(a)=b$. Then the quadruple system induced by (Q,f) is homogenous.*

Note that the converse is false, and the smallest example of this is obtained for order $v=16$ (see [8]).

Theorem. *Let (Q,f) be a finite Steiner 3-quasigroup. Then the mapping $\varphi(x)=f(s,t,x)$ is an automorphism of Q for each $s,t\in Q$, if and only if $f(a,b,f(u,v,w))=f(f(a,b,u),f(a,b,v),f(a,b,w))$ is an identity of Q.*

Proof. For every $a,b,x,y,z\in Q$ and the automorphism $\varphi(x)=f(a,b,x)$, we have

$$\begin{aligned}f(f(a,b,x),f(a,b,y),f(a,b,z))&=f(\varphi(x),\varphi(y),\varphi(z))\\&=\varphi(f(x,y,z))=f(a,b,f(x,y,z)).\end{aligned}$$

Conversely, let $f(a,b,f(u,v,w))=f(f(a,b,u),f(a,b,v),f(a,b,w))$ be an identity of Q, and s,t be arbitrary elements of Q. First we prove that the mapping $\varphi:Q\to Q$ defined by $\varphi(x)=f(s,t,x)$ is a bijection.

Since f is a quasigroup operation,

$$\varphi(x)=\varphi(y)\Rightarrow f(s,t,x)=f(s,t,y)\Rightarrow x=y,$$

which means that φ is injective.

Let $v \in Q$ and $u = f(s,t,v)$. Then

$$\varphi(u) = f(s,t,u) = f(s,t,f(s,t,v)) = v,$$

by the fact that (Q,f) is a Steiner 3-quasigroup. Hence, φ is surjective.

The mapping φ is a homomorphism, as a direct consequence of the identity. Namely, for $x,y,z \in Q$, we have

$$\begin{aligned}\varphi(f(x,y,z)) = f(s,t,f(x,y,z)) &= f(f(s,t,x), f(s,t,y), f(s,t,z)) \\ &= f(\varphi(x),\varphi(y),\varphi(z)).\end{aligned}$$

Corrolary2. *Let V be the variety of algebras with one ternary operation, defined by the identities*

$$\begin{aligned}&f(x,x,y) = y\\ &f(x,y,z) = f(x,z,y) = f(y,x,z)\\ &f(x,y,f(x,y,z)) = z\\ &f(a,b,f(x,y,z)) = f(f(a,b,x), f(a,b,y), f(a,b,z)).\end{aligned}$$

Then every finite algebra of V induces a homogenous SQS.

Proof. The first three of the defining identities of V determine the variety of Steiner 3-quasigroups, hence V is its subvariety.

Let (Q,f) be a finite algebra of V and (Q,B) be its induced quadruple system. We prove that for arbitrary elements $a,b \in Q$, the derived systems $(Q \setminus \{a\}, B_a)$ and $(Q \setminus \{a\}, B_b)$ of the quadruple system (Q,B) are isomorphic. Let $\varphi : Q \to Q$ be the mapping defined by $\varphi(x) = f(a,b,x)$. Then φ is an automorphism of Q by the preceding theorem. By the defining identities of V, we obtain $\varphi(a) = f(a,b,a) = b.$ Then by using the result of the Lemma, we get that the derived systems $(Q \setminus \{a\}, B_a)$ and $(Q \setminus \{a\}, B_b)$ are isomorphic.

3. EXAMPLES

The algebraic representative of the unique $SQS(8)$ satisfies the identity

$$f(a,b,f(x,y,z)) = f(f(a,b,x), f(a,b,y), f(a,b,z)) \qquad (1)$$

implying that the $SQS(8)$ is homogenous.

1 2 4 8	3 5 6 7
2 3 5 8	1 4 6 7
3 4 6 8	1 2 5 7
4 5 7 8	1 2 3 6
1 5 6 8	2 3 4 7
2 6 7 8	1 3 4 5
1 3 7 8	2 4 5 6

Figure 1: The unique $SQS(8)$

It is not necessary to check all the 8^5 quintuples to get this result. Namely, this SQS is a member of the class of Steiner quadruple systems of orders 2^n (constructed by Kirkman) whose corresponding Steiner 3-quasigroups satisfy the identity

$$f\left(x,y,f\left(z,y,t\right)\right)=f\left(f\left(x,y,z\right),y,t\right) \qquad (2)$$

The subvariety of the variety of Steiner 3-quasigroups which is determined by the above identity is its unique minimal subvariety, i.e. it is the unique atom in the lattice of all subvarieties of the variety of Steiner 3-quasigroups (see [10]). Its algebras can easily be obtained from the class of Boolean groups. Given a Boolean group $\left(S,+\right)$, one needs only to define a ternary operation f on S by

$$f\left(a,b,c\right)=a+b+c\,.$$

The main question which arises from this discussion is whether this minimal subvariety is a proper subvariety of the variety V of Corrolary 2, or the identities (1) and (2) are equivalent in the variety of Steiner 3-quasigroups.

The second “smallest” SQS is of order 10, and it is also unique (up to an isomorhism).

1 2 4 5	1 2 3 7	1 3 5 8
2 3 5 6	2 3 4 8	2 4 6 9
3 4 6 7	3 4 5 9	3 5 7 0
4 5 7 8	4 5 6 0	1 4 6 8
5 6 8 9	1 5 6 7	2 5 7 9
6 7 9 0	2 6 7 8	3 6 8 0
1 7 8 0	3 7 8 9	1 4 7 9
1 2 8 9	4 8 9 0	2 5 8 0
2 3 9 0	1 5 9 0	1 3 6 9
1 3 4 0	1 2 6 0	2 4 7 0

Figure 2: The unique $SQS(10)$

It is cyclic, which means that it has an automorphism consisting of a single cycle of length 10. Such an SQS belongs to a class of the so called transitive SQS, i.e. SQS whose automorphism group acts transitively on the elements. This class of SQS is precisely the class described in Corrolary 1. According to the Lemma, the $SQS(10)$ is an example of a homogenous SQS. However, (1) is not an identity of its corresponding Steiner 3-quasigroup:

$$f\left(1,2,f\left(4,7,9\right)\right)=f\left(1,2,1\right)=2,$$

but

$$f\left(f\left(1,2,4\right),f\left(1,2,7\right),f\left(1,2,9\right)\right)=f\left(5,3,8\right)=1.$$

This shows that the identity (1) provides sufficient, but not necessary condition for the induced SQS of a Steiner 3-quasigroup to be homogenous.

4. CONCLUSIONS

Steiner systems and other combinatorial designs have attracted mathematicians with their uniform distribution of elements into sets for a long time. Since the first results, a huge progress in combinatorics is made, but yet, there are so many unanswered questions and open problems.

The algebraic approach in studying Steiner systems contributed a lot in their understanding in the past 40 years, and the related algebraic structures have become a non-separating part of the combinatorial research.

The summation of the efforts in resolving the problem of classification of Steiner quadruple systems according to the number of their pairwise non-isomorphic derived triple systems is a collection of particular results which are partial and unsystematic. The research of SQS with a minimal, and SQS with a maximal possible number of classes of isomorphic DTS, i.e. homogenous and heterogenous SQS is not completed, as well. It was conjectured that both types of SQS exist, for every order $v \geq 16$.

The result of this paper brings a small contribution to the research of homogenous SQS, providing a nice description of a specific class of such SQS.

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ASYMMETRIC INNER PRODUCT AND THE ASYMMETRIC QUASI NORM FUNCTION

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Abstract. This paper attempts to generalize the semi scalar product concept according to G. Lumer by replacing Cauchy inequality with another inequality which is more generalized. Based on this attempt of generalization it is built a function which fulfils the conditions which are changed. In this paper it is also generalized quasi norm function by replacing homogeneity condition with a more restricted condition by producing this time a more generalized asymmetric semi norm function. As a result, in this paper it is defined the asymmetric inner product function and the asymmetric quasi norm function. Moreover, it is even given relation between these two.

1. INTRODUCTION

Semi-inner products, that can be naturally defined in general Banach spaces over the real or complex number field, play an important role in describing the geometric properties of these spaces.

Starting from its axiomatic, many researchers have made various modifications passing in its generalization. Semi-scalar products mark the very first generalizations of the scalar product function. The strong bond between these functions with the norm function has made it possible to obtain a lot of interesting results which are connected with the orthogonality and convexity [1],[2].

In [3],[4] it is also generalized the quasi norm function by replacing homogeneity condition with a more restricted condition by producing this time a more generalized asymmetric semi norm function.

Let be $p_0 : \mathbb{R} \to \mathbb{R}^+$ a function defined by:

$$p_0(x) = \begin{cases} |x|, & x < 0 \\ 2|x|, & x \ge 0 \end{cases}.$$

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Definition 1. *The* $p_0 : \mathbb{R} \to \mathbb{R}^+$ *function is called an asymmetric semi norm if:*

a) $p_0(x) \geq 0$ *for* $\forall x \in \mathbb{R}$.

b) $p_0(\lambda x) = \lambda p_0(x)$ *for* $\lambda > 0, \forall x \in \mathbb{R}$

c) $p_0(x+y) \leq p_0(x) + p_0(y)$ $\forall x, y \in \mathbb{R}$.

For every $x(x_1, x_2) \in \mathbb{R}^2$, we define the function $p(x) = p_0(x_1) + p_0(x_2)$, where $p_0(x_1), p_0(x_2)$ are asymmetric semi norms in $\mathbb{R}$.

Proposition 1. *The function* $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *such that* $p(x) = p_0(x_1) + p_0(x_2)$ *it is also an asymmetric semi norm in* $\mathbb{R}^2$.

Proof. a) We have

$$p(x) = p_0(x_1) + p_0(x_2) \geq 0, \ \forall (x_1, x_2) \in \mathbb{R}^2 \text{ and}$$
$$p(x) = 0 \Rightarrow p_0(x_1) = 0 \wedge p_0(x_2) = 0 \Rightarrow x_1 = x_2 = 0$$

b) We have

$$\begin{aligned} p(\lambda x) &= p_0(\lambda x_1) + p_0(\lambda x_2) = \lambda p_0(x_1) + \lambda p_0(x_2) \\ &= \lambda[p_0(x_1) + p_0(x_2)] = \lambda p(x), \quad \text{for } \lambda > 0. \end{aligned}$$

c) We have

$$\begin{aligned} p(x+y) &= p_0(x_1 + y_1) + p_0(x_2 + y_2) \\ &\leq p_0(x_1) + p_0(y_1) + p_0(x_2) + p_0(y_2) \\ &= [p_0(x_1) + p_0(x_2)] + [p_0(y_1) + p_0(y_2)] \\ &= p(x) + p(y). \end{aligned}$$

So $p(x+y) \leq p(x) + p(y)$, $\forall x, y \in \mathbb{R}^2$.

For every two points $x(x_1, x_2)$ and $y(y_1, y_2)$ in $\mathbb{R}^2$ we build the function $(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that:

$$(x, y) = \begin{cases} p(y)[\frac{x_1 y_1}{p_0(y_1)} + \frac{x_2 y_2}{p_0(y_2)}], & \text{for } y_1 \neq 0 \text{ and } y_2 \neq 0, \\ p(y)\frac{x_1 y_1}{p_0(y_1)}, & \text{for } y_1 \neq 0 \text{ and } y_2 = 0, \\ p(y)\frac{x_2 y_2}{p_0(y_2)}, & \text{for } y_1 = 0 \text{ and } y_2 \neq 0, \\ 0, & \text{for } y_1 = 0 \text{ and } y_2 = 0. \end{cases}.$$

The function defined above have the following properties:

1) $(x, x) \geq 0, \forall (x_1, x_2) \in \mathbb{R}^2$.

2) For $\lambda > 0$

$$(x, \lambda y) = p(\lambda y)[\frac{x_1(\lambda y_1)}{p_0(\lambda y_1)} + \frac{x_2(\lambda y_2)}{p_0(\lambda y_2)}] = \lambda^2 p(y)[\frac{x_1 y_1)}{\lambda p_0(y_1)} + \frac{x_2 y_2}{\lambda p_0(y_2)}]$$

$$= \lambda p(y)[\frac{x_1 y_1)}{p_0(y_1)} + \frac{x_2 y_2}{p_0(y_2)}].$$

$$(\lambda x, y) = p(y)[\frac{(\lambda x_1) y_1}{p_0(y_1)} + \frac{(\lambda x_2) y_2}{p_0(y_2)}] = \lambda p(y)[\frac{x_1 y_1}{p_0(y_1)} + \frac{x_2 y_2}{p_0(y_2)}] = \lambda (x, y),\ \lambda \in \mathbb{R}.$$

3) $(x + x', y) = (x, y) + (x', y)$

<u>*Case 1*</u>. $x = (x_1, x_2)$, $x' = (x_1', x_2')$ and $y = (y_1, y_2)$ where $x_1 \neq 0, x_2 \neq 0,$ $x_1' \neq 0,\ x_2' \neq 0$:

$$(x + x', y) = p(y)[\frac{(x_1 + x_1') y_1}{p_0(y_1)} + \frac{(x_2 + x_2') y_2}{p_0(y_2)}]$$

$$= p(y)[\frac{x_1 y_1}{p_0(y_1)} + \frac{x_1' y_1}{p_0(y_1)} + \frac{x_2 y_2}{p_0(y_2)} + \frac{x_2' y_2}{p_0(y_2)}]$$

$$= p(y)[\frac{x_1 y_1}{p_0(y_1)} + \frac{x_2 y_2}{p_0(y_2)}] + p(y)[\frac{x_1' y_1}{p_0(y_1)} + \frac{x_2' y_2}{p_0(y_2)}]$$

$$= (x, y) + (x', y).$$

<u>*Case 2*</u>. $x = (x_1, x_2)$, $x' = (x_1', 0)$ and $y = (y_1, y_2)$ where $x_1 \neq 0, x_2 \neq 0,$ $x_1' \neq 0$:

$$(x, y) = p(y)[\frac{x_1 y_1}{p_0(y_1)} + \frac{x_2 y_2}{p_0(y_2)}] \quad \textit{and} \quad (x', y) = p(y) \frac{x_1' y_1}{p_0(y_1)}$$

In this case $x + x' = (x_1 + x_1', x_2)$ therefore:

$$(x + x', y) = p(y)[\frac{(x_1 + x_1') y_1}{p_0(y_1)} + \frac{x_2 y_2}{p_0(y_2)}]$$

$$= p(y)[\frac{x_1 y_1}{p_0(y_1)} + \frac{x_2 y_2}{p_0(y_2)}] + p(y) \frac{x_1' y_1}{p_0(y_1)}$$

$$= (x, y) + (x', y).$$

The reconciliation $(x + x', y) = (x, y) + (x', y)$ goes equally in these cases:

a) $x = (x_1, x_2), x' = (0, x_2')$ and $y = (y_1, y_2)$ where $x_1 \neq 0, x_2 \neq 0, x_2' \neq 0$

b) $x = (x_1, 0), x' = (x_1', x_2')$ and $y = (y_1, y_2)$ where $x_1 \neq 0, x_1' \neq 0, x_2' \neq 0$

c) $x = (0, x_2), x' = (x_1', x_2')$ and $y = (y_1, y_2)$ where $x_2 \neq 0, x_1' \neq 0, x_2' \neq 0$

<u>*Case 3:*</u> $x = (x_1, x_2), x' = (x'_1, x'_2)$ and $y = (y_1, y_2)$ where $x_1 \neq 0, x_2 \neq 0,$ $x_1' \neq 0, x_2' \neq 0$ but $x_1 + x_1' = 0$ and $x_2 + x_2' = 0$ so $x_1 = -x_1'$ and $x_2 = -x_2'$.

In this case $x + x' = (0, 0)$ therefore $(x + x', y) = 0$ while:

$$(x', y) = p(y)[\frac{x_1' y_1}{p_0(y_1)} + \frac{x_2' y_2}{p_0(y_2)} = p(y)[\frac{-x_1 y_1}{p_0(y_1)} + \frac{-x_2 y_2}{p_0(y_2)}]$$

$$= -p(y)[\frac{x_1 y_1}{p_0(y_1)} + \frac{x_2 y_2}{p_0(y_2)}] = -(x, y)$$

from where: $(x, y) + (x', y) = 0 = (x + x', y)$.

Case 4: $x=(x_1,x_2)$, $x'=(x_1',x_2')$ and $y=(y_1,y_2)$ where $x_1\neq 0, x_2\neq 0$, $x_1'\neq 0$, $x_2'\neq 0$ but $x_1+x_1'=0$ so $x_1=-x_1'$.

In this case $x+x'=(0,x_2+x_2')$ therefore:

$$(x+x',y)=p(y)\frac{(x_2+x_2')y_2}{p_0(y_2)}=p(y)\frac{x_2y_2}{p_0(y_2)}+p(y)\frac{x_2'y_2}{p_0(y_2)}$$

while:

$$(x,y)=p(y)[\frac{x_1y_1}{p_0(y_1)}+\frac{x_2y_2}{p_0(y_2)}]$$

and

$$(x',y)=p(y)[\frac{x_1'y_1}{p_0(y_1)}+\frac{x_2'y_2}{p_0(y_2)}]=p(y)[\frac{-x_1y_1}{p_0(y_1)}+\frac{x_2'y_2}{p_0(y_2)}]$$

Since, from $(x,y)+(x',y)=p(y)\frac{x_2y_2}{p_0(y_2)}+p(y)\frac{x_2'y_2}{p_0(y_2)}=(x+x',y)$.

It is equally demonstrated when: $x=(x_1,x_2), x'=(x_1',x_2')$ and $y=(y_1,y_2)$ where $x_1\neq 0, x_2\neq 0, x_1'\neq 0, x_2'\neq 0$ but $x_2+x_2'=0$ so $x_2=-x_2'$.

4) From the definition of the function

$$p_0(x)=\begin{cases}|x|, & x<0\\ 2|x|, & x\geq 0\end{cases}$$

we obtain the inequality: $|x|\leq p_0(x)$, $\forall x\in\mathbb{R}^2$, from where:

$$|x_1|\leq p_0(x_1)\wedge|x_2|\leq p_0(x_2), |y_1|\leq p_0(y_1)\wedge|y_2|\leq p_0(y_2)$$

brings:

$$\begin{aligned}|(x,y)|&\leq p(y)[|x_1|\frac{|y_1|}{p_0(y_1)}+|x_2|\frac{|y_2|}{p_0(y_2)}]\\ &=p(y)[|x_1|+|x_2|]\leq p(y)\left[p_0(x_1)+p_0(x_2)\right]\\ &=p(y)p(x)=p(x)p(y).\end{aligned}$$

So $|(x,y)|\leq p(x)p(y)$, from where $(x,x)=|(x,x)|\leq p^2(x)$.

Remark. For (x,x) where $x(x_1,x_2)\in\mathbb{R}^2$ we have:

1) $x_1\neq 0, x_2\neq 0\Rightarrow p(x_1)\neq 0, p(x_2)\neq 0\Rightarrow$

$$\begin{aligned}(x,x)&=p(x)[\frac{x_1^2}{p_0(x_1)}+\frac{x_2^2}{p_0(x_2)}]=p(x)[\frac{|x_1|^2}{p_0(x_1)}+\frac{|x_2|^2}{p_0(x_2)}]\\ &\leq p(x)\left[p_0(x_1)+p_0(x_2)\right]=p^2(x).\end{aligned}$$

2) $x_1\neq 0, x_2=0\Rightarrow p(x_1)\neq 0, p(x_2)=0\Rightarrow$

$$(x,x)=p(x)\frac{x_1^2}{p_0(x_1)}=|x_1|^2\leq p_0^2(x)=p^2(x)$$

3) $x_1=0, x_2\neq 0\Rightarrow p(x_1)=0, p(x_2)\neq 0$

$$(x,x)=p(x)\frac{x_2^2}{p_0(x_2)}=|x_2|^2\le p_0^2(x)=p^2(x)$$

4) $x_1=0, x_2=0\Rightarrow p(x_1)=p(x_2)=0\Rightarrow p(x)=0\ \Rightarrow (x,x)=0=p^2(x)$

Finally: $|(x,x)|\le p^2(x)$.

Remark. Frankly, $(x,x)=p^2(x)$ every time is not true. Because for $x=(-1,2)$ we have $p(x)=|-1|+2|2|=5\Rightarrow p^2(x)=25$ and other side:

$$(x,x)=p(x)[\frac{x_1^2}{p_0(x_1)}+\frac{x_2^2}{p_0(x_2)}]=5[\frac{(-1)^2}{|-1|}+\frac{2^2}{2|2|}]=5[1+1]=10 .$$

In this case $(x,x)\ne p^2(x)$.

Record 1. *Also we can prove that* $p^2(x)\le 2(x,x)$.

Proof. <u>Case 1</u>: For $x(x_1,x_2)\in\mathbb{R}^2$ where $x_1<0\wedge x_2<0$ we have:

$$(x,x)=p(x)[\frac{x_1^2}{p_0(x_1)}+\frac{x_2^2}{p_0(x_2)}]=p(x)[\frac{|x_1|^2}{|x_1|}+\frac{|x_2|^2}{|x_2|}]=p(x)[|x_1|+|x_2|]=p(x)p(x)$$

or $p^2(x)=(x,x)\le 2(x,x)$

<u>Case 2</u>: For $x(x_1,x_2)\in\mathbb{R}^2$ where $x_1>0\wedge x_2>0$ we have:

$$(x,x)=p(x)[\frac{x_1^2}{p_0(x_1)}+\frac{x_2^2}{p_0(x_2)}]=p(x)[\frac{|x_1|^2}{2|x_1|}+\frac{|x_2|^2}{2|x_2|}]=p(x)[\frac{|x_1|}{2}+\frac{|x_2|}{2}]=\frac{p^2(x)}{2}.$$

So $p^2(x)=2(x,x)$.

<u>Case 3</u>: For $x(x_1,x_2)\in\mathbb{R}^2$ and $x_1>0\wedge x_2<0$ $[x_1<0\wedge x_2>0]$ we have:

$$(x,x)=p(x)[\frac{x_1^2}{p_0(x_1)}+\frac{x_2^2}{p_0(x_2)}]=p(x)[\frac{|x_1|^2}{2|x_1|}+\frac{|x_2|^2}{|x_2|}]$$

$$=p(x)[\frac{|x_1|}{2}+|x_2|]\ge p(x)[\frac{|x_1|}{2}+\frac{|x_2|}{2}]=\frac{p^2(x)}{2}.$$

So $p^2(x)\le 2(x,x)$.

Record 2. *The function* (x,y) *defined as above provides the benefit of the function* $\overline{p}:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *such that:* $\overline{p}(x)=\sqrt{(x,x)}$.

From the inequality: $p^2(x)\le 2(x,x)$ we have

$$p^2(x)\le 2\overline{p}^2(x)\ \text{or}\ p\ (x)\le\sqrt{2}\,\overline{p}\ (x),$$

and from the inequality $|(x,y)|\le p(x)p(y)$ we have:

$$|(x,y)|\le p(x)p(y)\le\sqrt{2}\,\overline{p}\ (x)\sqrt{2}\,\overline{p}(y)=2\overline{p}(x)\overline{p}(y).$$

So for the function $\overline{p}:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ these properties hold:

1) $\overline{p}(x) \geq 0, \overline{p}(x) = 0 \Rightarrow x = 0$ for $x \in \mathbb{R}^2$
2) $\overline{p}(\lambda x) = \lambda \overline{p}(x)$, for $\lambda > 0$, $x \in \mathbb{R}^2$
3) for $x, y \in \mathbb{R}^2$:

$$\begin{aligned}\overline{p}^2(x+y) &= |(x+y, x+y)| \leq |(x, x+y)| + |(y, x+y)| \\ &\leq 2\overline{p}(x)\overline{p}(x+y) + 2\overline{p}(y)\overline{p}(x+y) \\ &= 2\overline{p}(x+y)[\overline{p}(x) + \overline{p}(y)]\end{aligned}$$

So $\overline{p}(x+y) \leq 2[\overline{p}(x) + \overline{p}(y)]$, for $x, y \in \mathbb{R}^2$.

2. MAIN RESULTS

Definition 2. The function $(\cdot,\cdot): X \times X \to \mathbb{R}$, where X is a vectorial space, it is called *the asymmetric quasi inner product* if:

a) $(x, x) \geq 0$, $\forall x \in X$

b) $(\lambda x, y) = \lambda(x, y)$, $\forall (x, y) \in X^2$ and $\forall \lambda \in \mathbb{R}$

$(x, \lambda y) = \lambda(x, y)$, $\forall (x, y) \in X^2$ and $\lambda > 0$

c) $(x + x', y) = (x, y) + (x' + y)$, $\forall x, x', y \in X$

d) $|(x, y)|^2 \leq k(x, x)(y, y)$, for $k \geq 1$.

Definition 3. The function $p: X \to \mathbb{R}^+$ it is called *the asymmetric quasi norm function* if:

a) $p(x) \geq 0$, $\forall x \in X$

b) $p(\lambda x) = \lambda p(x)$, $\forall x \in X$ and $\lambda > 0$

c) $p(x + y) \leq k\left[p(x) + p(y)\right]$, $\forall (x, y) \in X^2$ and $k \geq 1$.

Proposition 2. *If* (x, y) *is the asymmetric quasi inner product function on* X *, then the function* $\overline{p}: X \to \mathbb{R}$ *such that* $\overline{p}(x) = \sqrt{(x, x)}$ *is an asymmetric quasi norm function.*

Proof. 1) We have

$$\overline{p}(x) = \sqrt{(x, x)} \geq 0, \forall x \in X$$

2) We have

$$\overline{p}(\lambda x) = \sqrt{(\lambda x, \lambda x)} = \sqrt{\lambda^2 (x, x)}, \text{ for } \lambda > 0.$$

Therefore

$$\overline{p}(\lambda x) = |\lambda| \sqrt{(x, x)} = \lambda \overline{p}(x).$$

3) We have

$$\begin{aligned}\overline{p}^2(x+y) &= |(x+y,x+y)| \\ &= |(x,x+y)+(y,x+y)| \\ &\le |(x,x+y)|+|(y,x+y)| \\ &\le \sqrt{k'(x,x)(x+y,x+y)}+\sqrt{k'(y,y)(x+y,x+y)} \\ &= \sqrt{k'}\,\overline{p}(x)\overline{p}(x+y)+\sqrt{k'}\,\overline{p}(y)\overline{p}(x+y) \\ &= \sqrt{k'}\left[\overline{p}(x)+\overline{p}(y)\right]\overline{p}(x+y).\end{aligned}$$

From where: $\overline{p}(x+y) \le \sqrt{k'}\left[\overline{p}(x)+\overline{p}(y)\right]$ and if we denote $\sqrt{k'} = k \ge 1$ we have:

$$\overline{p}(x+y) \le k[\overline{p}(x)+\overline{p}(y)].$$

3. CONCLUSIONS

An asymmetric quasi norm function can be obtained by an asymmetric inner product function, and the link between them is the function: $\overline{p}: X \to \mathbb{R}$, so that $\overline{p}(x) = \sqrt{(x,x)}$.

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